FLUID MECHANICS

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THIRD PRINTING

NEW YORK

D. VAN NOSTRAND COMPANY, Inc.

250 FOURTH AVENUE

First Published March 1941

Reprinted February 1942, August 1946

PREFACE

The present text has resulted from four years of experience in teaching fluid mechanics to many junior engineering students. These students majored in the aeronautical, electrical, mechanical and petroleum courses, and have had only the usual preparation in mathematics, mechanics and physics. The purpose of the course has been to better prepare the student for the problems to be encountered in the industrial field. Little time has been devoted to the abstract hydrodynamical reasoning, leaving the bulk of the time available for the practical study of the behavior of fluids. Concepts developed in the *hydraulics* field have been largely retained, but with much less emphasis being placed upon the purely empirical equations. The material has been expanded to include what might be termed the *industrial fluids*. To accomplish this goal, the study must include material covering both liquids and gases.

The text has been divided roughly into five parts, namely: *hydrostatics*, *measurement* of fluids, *transportation* of fluids in closed and open channels, *dynamics* of fluids and *centrifugal pumps*. We have found that students do not have well developed ideas concerning viscosity and dimensional analysis, so short chapters briefly covering these subjects have been introduced prior to the consideration of the different measuring devices. In this way, dimensional reasoning and the effect of viscosity could be considered throughout the remainder of the text wherever this seemed desirable.

It is felt that all of the material which is presented can be covered in an average three semester hour course. For those having less time at their disposal, the last two chapters can be omitted.

The authors wish to acknowledge the cooperation and encouragement offered by the many people with whom they have come in contact. Especially, they appreciate the encouragement given by L. J. Lassalle, Dean of the Engineering College, Louisiana State University. They also wish to thank and express their appreciation to C. E. McCashin, District Engineer, U. S. Geological Survey, for carefully reviewing the material on the flow in rivers, and to E. M. Kursheedt, Fairbanks-Morse and Company, for his kind interest and very careful review of the chapter on centrifugal pumps.

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February, 1941.

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FLUID MECHANICS

CHAPTER I

DEFINITIONS — PROPERTIES OF FLUIDS

1. Introduction. — **Fluid Mechanics** is that part of the science of mechanics which deals with the effect of forces on the form and motion of fluids. Fluid mechanics may be divided into hydrostatics and hydrokinetics.

Hydrostatics is the study of fluids at rest. It is an exact science whose laws may be derived mathematically without recourse to a great amount of experimentation other than that required for the determination of the weights of the fluids.

Hydrokinetics is the study of fluids in motion.

While the mechanics of rigid bodies considers the motion of bodies whose particles remain sensibly fixed relative to each other, and strength of materials is largely a study of the small relative displacements of particles produced by external forces acting on semi-rigid solids, hydrokinetics must concern itself with bodies whose particles move widely relative to each other, and have paths not easily traceable except in a few ideal cases.

Because the motion of fluids is so complicated, very few of the laws pertaining to this phase of fluid mechanics have been obtained by direct mathematical deduction. It has always been necessary to resort to experiment.

Until recently, this experimental work was confined almost entirely to the study of the motion of water. Out of this grew the science of hydraulics with a mass of empirical equations which were limited in use because they applied mainly to the behavior of water and then only over a limited range.

The great advances in industry, especially in the chemical, mechanical and aeronautical fields, have been accompanied by an ever increasing use of gases and liquids other than water. It became necessary to learn how these fluids behaved when in motion. The need for this knowledge furnished the incentive for many investigators whose work demonstrated that although a complete mathematical treatment of the subject was impossible, certain criteria for similarity of flow existed which permitted the results of an experiment with one fluid to be used in predicting the behavior

of an entirely different fluid under similar circumstances. It was the recognition of the existence of similarity of flow and the determination of the criteria for similarity which brought order to the science of Fluid Mechanics and enabled its laws to be generalized so as to apply to the motion of the fluids which are encountered in industry.

2. Fluids. — Fluids are divided into two main classes, namely: liquids and gases. A liquid is any substance which will take the shape of an enclosing container, but need not completely fill it; while a gas is any substance which will not only take the shape of the enclosing container, but will also fill it regardless of the quantity enclosed.

When fluids flow, relative motion between the particles of the fluid takes place; and the rate at which this relative motion occurs depends upon a property of fluids called *viscosity*. Although a mathematical definition of viscosity will not appear until a later chapter (see Chap. IV, p. 46), it is sufficient for our present purposes to realize that all fluids do have this property, and that it is in the nature of an internal frictional resistance between the particles of the fluid.

Some fluids exhibit this property to a greater extent than others. The thicker grades of oil and molasses, for example, will not pour as readily as water, while the viscosity of gases is even less than that of water. A fluid in which the molecules offered no shearing resistance whatsoever to relative motion would be called a *perfect fluid*. No such fluid exists, but it is often assumed in order to simplify the mathematical treatment of certain types of flow.

Certain liquids are so viscous that it is sometimes difficult to distinguish them from solids. Pitch is such a liquid. Should a sudden force be applied to pitch, such as a blow from a hammer, it would shatter just as would any other brittle, solid substance, but if a gradual force be applied for a considerable length of time, it would appear fluid. A piece of pitch can be placed on a flat surface and, due to the force of its own weight, it will flow and cover a large area. Substances of this type are *thick* liquids as compared to water which is a *thin* liquid.

A fluid may pass from one state into another. Using water as an example, we are familiar with it as a solid, liquid, or gas in the form of ice, water, or steam. This ability to exist in several different states is common to many substances.

3. Absolute Pressure and Absolute Temperature. — The intensity of pressure on a fluid is often referred to some arbitrary base. In many cases, atmospheric pressure is used as this base. The pressure of the atmosphere, however, varies from day to day and from hour to hour; moreover, it is not the same at points having different elevations above mean sea level. These facts are demonstrated by the fluctuating baro-

metric readings which are recorded by the Weather Bureau stations of this country.

For problems involving gases, it is necessary to know the pressure with reference to a non-changing base. Consequently, *absolute pressures* are employed. The term " absolute pressure " is used to indicate the pressure above a perfect vacuum. It is the pressure of the atmosphere in addition to the pressure as recorded by most ordinary gages. At sea level, the atmospheric pressure is taken equal to 14.7 lb. per sq. in. which is equivalent to a barometric pressure of 760 millimeters of mercury. Unless otherwise stated, absolute pressure will be taken as 14.7 lb. per sq. in. more than the pressure as read on most gages.

Just as pressures may be referred to different bases, temperatures are also given on different scales which do not have the same reference or zero point. Thus we have the centigrade scale with the temperature at which water freezes taken as zero. On the Fahrenheit scale, this same temperature corresponds to a reading of 32°. A 1° temperature change on the centigrade scale corresponds to a 1.8° change on the Fahrenheit scale.

Gases behave in such a way that it is necessary to refer their temperature to *absolute zero*. Zero absolute temperature is that temperature at which all molecular activity is considered to cease. This temperature is 273.17° below zero on the centigrade scale, or 459.6° below zero on the Fahrenheit scale. For all problem work in this text, 273° C. or 460° F., respectively, will be used for these values. Thus temperature in degrees absolute is equal to temperature in degrees Fahrenheit + 460 on the Fahrenheit scale, and to temperature in degrees centigrade + 273 on the centigrade scale. The student should not neglect to make these conversions in the solution of problems involving gases.

A gas is said to be at *standard conditions* when it is at a temperature of 0° C., and under a pressure equal to that of the atmosphere at sea level, or 760 mm. of mercury. This is the definition used in physics and chemistry.

4. Effect of Change of Pressure on Liquids. — Within the limits of ordinary engineering problems, the error introduced by neglecting the change in volume due to a change in pressure is usually not excessive. If we imagine the pressure on the faces of a cube of liquid to change by a small amount, the ratio of the change in unit pressure to the change in volume per unit volume is called the *bulk modulus of elasticity*. Expressed mathematically

$$K = -\frac{\Delta p}{\dfrac{\Delta V}{V}} = -\frac{V\Delta p}{\Delta V} \tag{1}$$

in which K is the bulk modulus of elasticity; ΔV is a small change in vol-

ume; and Δp is a small change in unit pressure. The minus sign indicates that the volume decreases as the unit pressure is increased. Since the ratio of the change in volume to the original volume is a pure number, the unit for K is the same as that for unit pressure; i.e., pounds per square inch or pounds per square foot.

The value of the bulk modulus for liquids varies, depending upon the temperature and pressure at which it is measured. This variation for water is not large at low pressures and temperatures, and the modulus for water may be taken equal to 300,000 lb. per sq. in. for temperatures up to 200° F., and pressures up to 4,000 lb. per sq. in. The bulk modulus for steel is about 20,000,000 lb. per sq. in., so that the volume of water changes approximately seventy times as much as steel for the same increment in pressure. Even this high relative compressibility, however, does not require that the change in volume and density of water due to pressure changes be considered in ordinary engineering computations. The density of water one-half mile below the surface of the ocean, for instance, is only one-half of one per cent greater than that at the surface. Other liquids behave in more or less the same manner and no large errors will result from considering them incompressible.

5. Effect of Change of Temperature on Liquids. — Temperature changes in liquids are important especially because of their effects on the density, viscosity and gas dissolving properties of the liquids.

In general, a rise in the temperature of a liquid will produce an increase in volume and a decrease in density. Table I gives the weight of pure water at different temperatures at atmospheric pressure.

TABLE I. VARIATION OF WEIGHT OF PURE WATER

Temperature °F.	*Pounds per Cu. Ft.*	*Temperature °F.*	*Pounds per Cu. Ft.*	*Temperature °F.*	*Pounds per Cu. Ft.*
32	62.42	100	62.00	170	60.80
40	62.42	110	61.86	180	60.59
50	62.41	120	61.72	190	60.36
60	62.37	130	61.56	200	60.14
70	62.30	140	61.39	210	59.89
80	62.22	150	61.20	212	59.84
90	62.12	160	61.01	...	.....

Water has its maximum density at a temperature of 4° C., and at this temperature it weighs 62.424 lb. per cu. ft. When no temperature is given in the statement of a problem in this text, the weight of water will be assumed equal to 62.4 lb. per cu. ft.

The viscosity of liquids decreases so rapidly with increasing temperature that it will always be necessary to consider the temperature in the solution

of problems involving this property. More often, the density as well as the viscosity of the liquid must be known, so that the effect of temperature on both of these properties should be taken into consideration.

At higher temperatures, liquids give up entrapped gases very readily; this is especially true whenever a partial vacuum exists. Because of this occurrence, real difficulty is encountered in pumping hot water by means of the centrifugal type pump. The enclosed gases are released, and cause the pump to lose its prime.

PROBLEMS

1. When the unit of pressure on 1 cu. ft. of water at 32° F. is changed from 1000 lb. per sq. in. to 7000 lb. per sq. in., the volume changes from 1.000 cu. ft. to 0.9795 cu. ft. With these data given, compute the bulk modulus of elasticity.

2. Water at the base of the nozzle for a Pelton water wheel is under a pressure head of 2500 ft. This corresponds to a pressure of approximately 1085 lb. per sq. in. gage. At the same temperature and atmospheric pressure the water weighs 62.424 lb. per cu. ft. (*a*) If the bulk modulus is 300,000 lb. per sq. in., find the weight of a cubic foot of water at the base of the nozzle. (*b*) What per cent error would result in neglecting compressibility?

Ans. (*a*) w = 62.650 lb. per cu. ft. (*b*) 0.36 per cent.

3. The temperature of water at atmospheric pressure is changed from 100° to 90° F. What increase in pressure would be required to cause a change in volume equal to that produced by the temperature change? Assume 315,000 lb. per sq. in. for the bulk modulus.

6. Effect of Temperature and Pressure on Gases. — The Gas Laws. — A complete treatment of this subject would carry us too deeply into the study of thermodynamics. Consequently, only the simplest and most essential relationships between volumes, pressures, and temperatures of gases will be considered. If the student desires more information than is given here, he should consult one of the standard texts on thermodynamics.

The effects produced by changes of pressure and temperature on gases are much greater and more significant than those produced on liquids. Moreover, a change in pressure may not only change the volume, but may also increase or decrease the temperature appreciably. The same statement can be made with regard to temperature changes in altering volume and pressure.

The volume of a given quantity of gas decreases as the pressure on the gas is increased; at the same time, unless heat is rejected through the walls of the container, the gas temperature will rise. When the rate at which heat is removed from the gas is such that the temperature remains constant, the process is called an *isothermal* compression. On the other hand, an isothermal expansion is one in which the pressure is decreased, and heat is added to the gas in order to maintain its temperature constant. In either case, the amount of heat added or removed represents a definite amount of energy.

The energy which is possessed by a gas because of the kinetic energy of its molecules is called its intrinsic or internal energy. According to the molecular theory of gases this energy for a perfect gas is directly proportional to its absolute temperature. Since in an isothermal process the temperature remains constant, the intrinsic energy of the gas at the beginning and at the end of the process must be the same. When heat is added to a gas isothermally, the gas expands and does work against the resisting force of the pressure. The energy required to do this must be exactly equal to the heat added during the process because, as stated above, no energy is removed from the gas itself.

The relationship between the pressures and volumes at different stages of an isothermal process is given by Mariotte's or Boyle's Law, which states that the density of a given quantity of gas is proportional to the absolute pressure; or that the volume of the same quantity of gas is inversely proportional to the absolute pressure, provided that the temperature remains constant. This may be expressed mathematically by the equation

$$p_1 v_1 = p_2 v_2 \quad \text{or} \quad pv = \text{Constant} \tag{2}$$

In this equation, p_1 and p_2 are the initial and final unit absolute pressures, and v_1 and v_2 the initial and final volumes. The equation is used in finding an unknown pressure or volume after an isothermal compression or expansion when the other three terms in the equation are known.

Equation (2) may be written as

$$\frac{p_1}{p_2} \quad \frac{v_2}{v_1}$$

It is now in the form of a ratio, and shows that correct results will be obtained from its use as long as the units of the pressure intensities are the same, and those for the volumes are the same. For example, the absolute pressures may be expressed in pounds per square inch, pounds per square foot, atmospheres, or in any other convenient unit for pressures. The volumes are generally expressed in cubic feet. In general, it is best to use consistent units; i.e., p in pounds per square foot and v in cubic feet.

When the equation is used in connection with one pound of gas, v is the volume of one pound of gas, and is called the *specific volume.* The specific volume of a gas is equal to the reciprocal of the *specific weight* or weight per cubic foot. Thus

$$v = \frac{1}{w}$$

where v = specific volume in cubic feet per pound,
w = specific weight in pounds per cubic foot.

Equation (2) is true only for a *perfect gas*. In thermodynamics, a gas is said to be perfect only when it follows Boyle's Law. The student will notice that this definition of a perfect gas differs from that of a perfect fluid as given on p. 2. There, a perfect fluid was defined as one in which the particles offered no resistance whatsoever to relative motion. No confusion will arise if we realize that one definition is given from the standpoint of thermodynamics and the other from the standpoint of fluid mechanics; in other words, a gas may follow Boyle's Law very closely and yet be far from a frictionless fluid.

No gas follows Boyle's Law exactly, but many gases do so very closely. The deviation is not the same for different gases, and the amount of this deviation is a function of the difference between the temperature of the gas and the temperature at which it would condense into a liquid. Vapors, such as steam, are relatively nearer to their condensation temperatures than are the so called permanent gases, such as hydrogen and oxygen. The changes in pressure and volume during an expansion or contraction of steam cannot be predicted very accurately by the use of Boyle's Law, while these changes for the permanent gases follow it quite closely. Most of the problems in this text will be such that Boyle's Law will apply with little error.

If heat is added to a given mass of gas in a closed vessel of constant volume, the temperature of the gas will rise; and the pressure exerted by the sides of the vessel on the gas will increase. Should the vessel be a cylinder, fitted with a piston to which a constant force is applied, the temperature would again rise, but the gas would expand at constant pressure. The heat energy added to the gas would then be sufficient to raise the temperature of the gas, and to do work on the piston.

The relationship between temperatures and volumes during an expansion or contraction at constant pressure is given by Gay-Lussac's, or Charles' Law, which states that the volumes are directly proportional to the absolute temperatures. Mathematically, this is

$$\frac{v_1}{v_2} = \frac{T_1}{T_2} \tag{3}$$

in which v_1 and v_2 are the initial and final volumes of the gas, and T_1 and T_2 are the initial and final absolute temperatures.

A gas may undergo changes of volume due to simultaneous changes of pressure and temperature. The law followed is derived by combining equations (2) and (3) in the following manner. The original condition of the gas is defined by the initial pressure volume and temperature p_1, v_1, and T_1. After the change takes place, these same quantities become p_2, v_2, and T_2. One pound of gas will be assumed so that v is the specific vol-

ume. The change may be considered to occur in two steps. First, suppose the gas to expand at constant temperature from p_1, v_1, T_1 to an intermediate stage represented by p_2, v_x, T_1. Applying Boyle's Law, we have

$$p_1v_1 = p_2v_x$$

and

$$v_x = \frac{p_1v_1}{p_2} \tag{a}$$

The temperature of the gas here is still T_1. The gas is then assumed to expand at constant pressure from p_2, v_x, T_1 to p_2, v_2, T_2. Applying Gay-Lussac's Law, we have

$$T_1 \qquad T_2$$

and

$$v_x = \frac{v_2T_1}{T_2} \tag{b}$$

Equating (a) and (b) results in

$$\frac{p_1v_1}{p_2} = \frac{v_2T_1}{T_2}$$

or

$$\frac{p_1v_1}{T_1} = \frac{p_2v_2}{T_2} = R \tag{c}$$

from which we obtain the general expression

$$pv = RT \tag{4}$$

where p = pressure in pounds per square foot,
v = specific volume in cubic feet per pound,
T = absolute temperature,
R = the gas constant for 1 lb. of gas.

If simultaneous values of pressure, volume, and temperature for 1 lb. of gas are known, R may be computed by Eq. (c) of this article. For example, air at standard conditions (p = 14.7 lb. per sq. in., T = 32° F., v = 12.38 cu. ft. per lb.) would have

$$R = \frac{pv}{T} = \frac{14.7 \times 144 \times 12.38}{32 + 460} = 53.3$$

The value of R has been computed in the above manner for many of the common gases. Some values of R are given in Table II, p. 9.

If the value of R is known, Eq. (4) can be used for finding an unknown pressure, volume or temperature when two of these quantities are given.

7. Adiabatic Process. — It was seen in Art. 6, p. 5, that an isothermal expansion or contraction could be made to take place by a suitable addition or subtraction of heat. An *adiabatic process* differs from this in that the expansion or contraction takes place without the addition or subtraction of heat. Space does not permit a complete derivation of the law followed in this kind of a process, but it may be shown to be

$$p_1v_1^n = p_2v_2^n \quad \text{or} \quad pv^n = \text{Constant} \tag{5}$$

In this equation p_1 and p_2 are the initial and final pressures; v_1 and v_2 are the initial and final specific volumes; and n is the ratio of the specific heat at constant pressure to the specific heat at constant volume, as defined below. Pertinent constants for some gases are given in Table II.

TABLE II. SPECIFIC HEATS AND CONSTANTS FOR GASES AT STANDARD CONDITIONS

Gas	C_p *B.t.u. per (lb.)(°F.)*	C_v *B.t.u. per (lb.)(°F.)*	n	R	v *cu. ft./lb.*	w *lb./cu. ft.*
Air	0.242	0.173	1.40	53.3	12.39	0.0807
Helium	1.25	0.752	1.66	386.6	90.0	0.0111
Hydrogen	3.41	2.42	1.46	772.0	177.1	0.00565
Methane	0.504	0.376	1.31	96.5	22.34	0.0448
Nitrogen	0.247	0.176	1.40	55.2	12.81	0.0781
Oxygen	0.217	0.155	1.40	48.3	11.22	0.0891

Since the gas, in expanding, does work against a constantly decreasing pressure, and since no energy is supplied to the gas from an outside source, the work must be performed at the expense of the intrinsic energy of the gas itself. This means that the temperature of the gas will be lower after the expansion takes place. Conversely, if a gas is compressed adiabatically, the work done on the gas will be stored in the gas as intrinsic energy and the temperature will rise. The temperature at any point of an adiabatic process may be found by the use of Eq. (4), p. 8, if it is required.

PROBLEMS

4. A tank has a capacity of 40 cu. ft. and contains air at atmospheric pressure. (*a*) How many cubic feet of air at atmospheric pressure must be pumped into the tank in order to attain a pressure of 10 atmospheres (abs.)? Assume that the temperature does not change. (*b*) If the temperature is 70° F., what weight of air would this represent? $R = 53.3$.

Ans. (*a*) 360 cu. ft., (*b*) $W = 26.9$ lb.

5. A rubber balloon is filled with 10 cu. ft. of hydrogen at a pressure of 5 lb. per sq. in. and temperature of 60° F. When the temperature of the gas is increased to 110° F., the volume occupied by the balloon becomes 15 cu. ft. Find the new pressure.

6. An automobile tire has a capacity of 1 cu. ft. The tire is filled with air at 65° F. under a pressure of 30 lb. per sq. in. (*a*) What weight of air does the tire

contain? (*b*) Assuming the volume to remain constant, what is the tire pressure when the tire is heated to 100° F.?

7. (*a*) What is the specific weight of helium at a pressure of 60 lb. per sq. in. gage and a temperature of 70° F.? $R = 386$. (*b*) At what temperature will helium weigh 0.09 lb. per cu. ft. if the pressure is 100 lb. per sq. in. gage?

Ans. (*a*) $w = 0.0526$ lb. per cu. ft., (*b*) $T = 16°$ F.

8. Specific Heats of Gases. — The *specific heat* of a gas at a given initial temperature is the amount of heat required to raise the temperature of 1 lb. of gas 1° F. The heat may be added while the volume of the gas remains constant or while the pressure on the gas remains constant. Hence there may be two values for the specific heat.

The specific heat at constant volume (C_v) is defined as the amount of heat required to raise the temperature of 1 lb. of gas 1° F., the volume remaining constant.

The specific heat at constant pressure (C_p) is the amount of heat required to raise the temperature of 1 lb. of gas 1° F., the pressure remaining constant.

Since the heat energy added at constant pressure is employed partly in raising the temperature and partly in doing work against the pressure as expansion takes place, C_p is greater than C_v. Values of C_p and C_v are given in Table II, p. 9.

The ratio of the specific heats is

$$n = \frac{C_p}{C_v}$$

The unit for these specific heats is the British Thermal Unit (B.t.u.). The B.t.u. is mechanically equivalent to 778 ft. lb. of energy, so that

$$1 \text{ B.t.u.} = 778 \text{ ft. lb.}$$

The specific heats of gases are constant over large ranges of temperature. This is not so with vapors or gases near their condensation temperature, so that Eq. (4) cannot be used when this type of fluid is involved.

PROBLEMS

8. Three cubic feet of air at a temperature of 80° F. expands adiabatically from a pressure of 60 lb. per sq. in. to 35 lb. per sq. in. abs. (*a*) Find the volume after the expansion. (*b*) What is the temperature of the air after the expansion? $n = 1.4$. $R = 53.3$. (*c*) What supplies the energy for the work of expansion against the surrounding pressure?

9. Oxygen in passing through a venturi tube expands adiabatically from a gage pressure of 100 lb. per sq. in. at the inlet of the meter to 80 lb. per sq. in. at the throat. The temperature at the inlet is 50° F. Determine (*a*) the specific weight of the gas at the inlet; (*b*) the specific weight at the throat; (*c*) the temperature at the throat. $n = 1.4$. $R = 48.3$.

Ans. (*a*) $w_1 = 0.67$ lb. per cu. ft., (*b*) $w_2 = 0.585$ lb. per cu. ft., (*c*) $T = 23°$ F.

9. Surface Tension. — It is a well-known fact that the surface of contact between a liquid and a gas or between two immiscible liquids, when deformed slightly, acts very much like a film capable of resisting a small tensile stress. The explanation for this lies in a consideration of the cohesive forces between molecules at such a boundary. Taking, for example, the surface between water and air, the molecules near the surface are acted upon by the water molecules immediately below them and by the air molecules immediately above. Since the force exerted by the former is the greater, there is a tendency for the surface molecules to be drawn down into the interior, and for the surface to contract to a minimum area. Any small deformation of this surface represents a deviation from the condition of minimum area and induces stresses in the surface similar to those in an elastic membrane or film when a lateral force is applied.

We have seen that a surface of the type described above acts very much like an elastic film. For convenience, we shall assume that such a film exists. Imagine a line to be drawn in the surface and visualize the force exerted by the portions of the film on each other. This attractive force gives rise to the phenomenon known as surface tension.

The magnitude of this force expressed as a force per unit length of line is the quantitative measure of *surface tension.* It depends on the liquid and gas, or the liquids in contact, and decreases with a rise in temperature, as might have been concluded from the fact that the cohesive attraction of the molecules of a liquid is reduced by an increase in temperature. For water in contact with air, surface tension decreases almost uniformly from approximately 0.005 lb. per ft. at 32° F. to 0.004 lb. per ft. at 212° F. For mercury and water at ordinary temperatures it is about 0.027 lb. per ft.

A surface film will assume a curved shape when pressures on one side in a normal direction exceed those on the other in a normal direction. The relationship between this difference in pressures and surface tension is important to our discussion of capillarity which is to follow; so we will determine it for the simple case where the film takes a spherical shape.

Let Fig. 1 represent such a spherical surface of radius, R, with p_1 the greater radial unit pressure and p_2 the smaller. T represents the surface tension in force per unit length along the rim of the spherical segment. The total force acting upward in the Y direction can be shown to be equal to the difference in unit pressures multiplied by the projected area of the spherical segment on a plane perpendicular to the Y direction (see Chap.

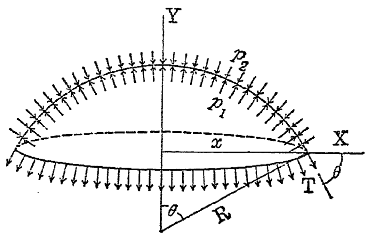


FIG. 1.

III, p. 45). Summing forces along the Y axis, therefore, we have

$$(p_1 - p_2)\pi x^2 - 2\pi x T \sin\theta = 0$$

$$\sin\theta = \frac{x}{R}$$

$$(p_1 - p_2)\pi x^2 = \frac{2\pi x^2 T}{R}$$

or

$$T = \frac{(p_1 - p_2)R}{2} \tag{6}$$

If p is the difference between the unit pressures on opposite sides of the surface

$$T = \frac{pR}{2} \quad \text{or} \quad p = \frac{2T}{R} \tag{7}$$

In these equations T is in pounds per foot, R is in feet, and p is in pounds per square foot.

The student should bear in mind that these equations are applicable only when the surface is known to be spherical.

PROBLEMS

10. In an experiment for determining the surface tension of water, the difference in unit pressure on opposite sides of a spherical water-air surface is known to be 0.012 lb. per sq. in. The radius of the surface is 0.06 in. Determine the value of surface tension.

11. Derive the relationship between surface tension and the difference in unit pressures on opposite sides of a meniscus which assumes a cylindrical shape.

Ans. $T = pR$.

12. How much greater than atmospheric is the unit pressure on the inside surface of a cylindrical jet of water $\frac{1}{4}$ in. in diameter, due to surface tension? Take surface tension equal to 0.005 lb. per ft. See answer to Prob. 11.

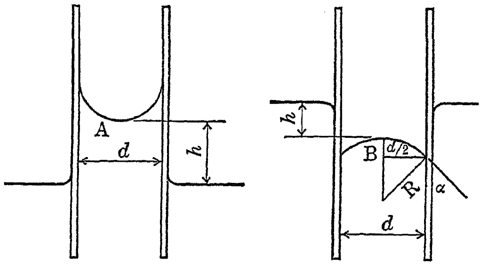
10. Capillarity. — When a liquid is in contact with a solid the attraction of the molecules of the solid for those of the liquid may be greater than that between the molecules of the liquid itself. Adhesion is then said to be greater than cohesion and the liquid wets the solid. Water wets glass while mercury does not, so that in the latter case cohesion between molecules of mercury is greater than adhesion between mercury and glass.

These properties of adhesion and cohesion, in addition to surface tension, result in the phenomenon of capillarity. If a tube of small bore is placed in water as shown in Fig. 2*a*, the water will rise above the surrounding liquid. On the other hand, if the tube penetrates a mercury surface (Fig. 2*b*) the mercury in the tube will be depressed.

Because of adhesion, the line of contact between the water inside of the tube and the glass rises relative both to the water surrounding the tube

and to the water in the center of the tube. If the tube is small, the meniscus, which is the fluid surface within the tube, may be assumed to be spherical and its radius computed provided that the angle of contact between the meniscus and tube is known. This angle for water and glass is 0° and for mercury and glass is approximately 53°. Taking d as the diameter of the tube and α the angle of contact, the radius of the meniscus is

$$R = \frac{d}{2 \cos \alpha},$$



(a) (b)

Fig. 2. Meniscus for (a) water and (b) mercury.

and the difference in pressures on opposite sides of the surface in the tube is

$$p = \frac{2T}{R} = \frac{4T \cos \alpha}{d},$$

where T is surface tension as before. The unit pressure at A is less than atmospheric and that at B greater than atmospheric by this amount.

Noting that the pressure on the surface surrounding the tube is atmospheric, the average amount of capillary elevation in the case of water, or depression in the case of mercury, above or below the surrounding fluid is

$$h = \frac{p}{w}$$ See Chap. II, p. 18.

or $$h = \frac{4T \cos \alpha}{wd} \tag{8}$$

where h = capillary rise or depression in feet,
T = surface tension in pounds per foot,
w = specific weight of liquid in pounds per cubic foot,
d = diameter of tube in feet,
α = angle of contact of meniscus.

It will be seen that the elevation of the liquid in a glass tube above some point in the liquid can be used to measure the "intensity of pressure" at that point. For tubes $\frac{1}{8}$ in. in diameter or less, the equation derived above for capillary correction may be used, but it is not applicable for larger tubes. However, the correction becomes smaller as the diameter increases, and where water or mercury are used as measuring liquids, a tube $\frac{1}{2}$ in. in diameter or larger will obviate any need for a correction. The correction

for tubes of intermediate size is uncertain, and use of such tubes is discouraged when great accuracy is required.

PROBLEMS

13. The unit pressure at a point in a liquid is measured by a column of water in a tube $\frac{1}{16}$ in. in diameter. What correction should be applied to the reading because of capillarity? Take surface tension equal to 0.0045 lb. per ft. and the angle of contact of the meniscus as zero degrees.

Ans. -0.055 ft.

14. Two glass plates, 0.125 in. apart, dip vertically into a vessel of water. At what height will the water between the plates stand above the surface level of the water in the vessel? Note that the meniscus here is cylindrical. Take the surface tension of water to be 0.005 lb. per ft.

11. Notation. — As far as possible, the notation as prepared by the Special Committee on Irrigation Hydraulics of the American Society of Civil Engineers and published as *Manual of Engineering Practice — No. 11*, entitled "Letter Symbols and Glossary for Hydraulics with Special Reference to Irrigation" will be used. Wherever this is not possible, nomenclature which is in common usage will be adopted and it will be strictly defined. There will be little likelihood of misunderstanding.

12. Units. — In the main, the foot, pound, second system of units will be used, where pound is the unit of force. This is not always desirable, but every attempt will be made to make the changes in units clear. Practice makes it desirable that other systems be used at times. For example, the diameter of a pipe is given in inches and the viscosity of a fluid is generally given in poises when the viscosity is expressed in the c.g.s. system.

The value of acceleration due to gravity will come into certain problems. The value of the acceleration (g) varies with the elevation and with latitude, but this variation will not be considered in the different problems. A value of 32.16 ft. per sec. per sec. will be used in the solution of the problems.

The units of any result are determined by the units of the quantities which appear together in the equation. For example, the quantity of water flowing in a pipe, or the discharge, is given by the equation

$$Q = AV$$

where A is cross-sectional area and V is the average velocity of the water.

If the cross-sectional area of the stream is expressed in square feet and the average velocity of the water in feet per second, then the unit for discharge will be

$$\text{ft.}^2 \times \frac{\text{ft.}}{\text{sec.}} = \frac{\text{ft.}^3}{\text{sec.}} \quad \text{or} \quad \text{cu. ft. per sec.}$$

This same principle must hold true for any formula which is dimensionally correct.

CHAPTER II

PRESSURE IN FLUIDS

13. Pressure at a Point in a Fluid at Rest. — A fluid may be considered to be made up of a large number of particles, each particle being in contact with other similar particles. Figure 3 shows a cubical particle in a mass of fluid at rest. The only forces holding this element in equilibrium are the normal forces exerted by adjacent particles on the faces of the cube and the weight force. The particle exerts forces, equal and opposite to the ones shown, on the particles which touch it. No tangential forces act on the faces of the cube because if such forces existed the element would undergo a distortion or change of shape. As a matter of fact, a fluid might be defined as a substance within which no tangential or frictional forces exist when at rest. Now if the element is assumed to become smaller and smaller the lengths of the sides eventually become infinitesimal and the weight of the particle becomes negligible in comparison to the forces holding it in equilibrium. The force acting on the particle in any direction divided by the small element of area is then called the intensity of pressure, or pressure per unit area at the point. This may be written as

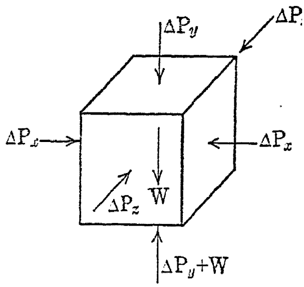


FIG. 3.

$$p = \frac{dP}{dA} \qquad (9)$$

If dP is a force in pounds, and dA an area in square feet, the intensity of pressure, p, will be in pounds per square foot; if the area is in square inches, p will be in pounds per square inch.

The discussion of intensity of pressure given above serves to show that there is a pressure or force exerted between particles at any point in a fluid. If a group of particles similar to the one described above occupy a finite area upon which the intensity of pressure is the same, then the intensity of pressure is equal to the total force, P, divided by the area, or

$$p = \frac{P}{\ } \qquad (10)$$

In this equation p is the intensity of pressure in pounds per square inch or pounds per square foot, depending upon the units for A. P is the total force acting on the area in pounds, and A is the area in square inches or square feet. Both p and P are in a direction normal to the area.

Very often the term *pressure* is erroneously used in the place of intensity of pressure. The student should have little trouble distinguishing between a unit pressure and a total pressure, especially when units are given.

14. At Any Point within a Fluid at Rest, the Intensity of Pressure Is the Same in All Directions. — Figure 4 represents the cross section of a small triangular prism of fluid of differential dimensions. The prism is dx in width, dy in height, and of unit length perpendicular to the plane of the paper. The intensities of pressures on the faces perpendicular to the paper are as shown.

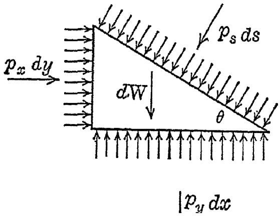


Fig. 4.

The prism is in equilibrium due to the total forces acting on its faces and its weight dW. Using the equation $\sum F_x = 0$, we obtain for the x direction,

$$p_x dy = p_s ds \sin\theta$$

But $$\sin\theta = \frac{dy}{ds}$$

Substituting this value for $\sin\theta$ in the equation above,

$$p_x dy = p_s ds \frac{dy}{ds},$$

or $$p_x = p_s$$

Using the equation $\sum F_y = 0$,

$$p_y dx = p_s ds \cos\theta + dW$$

$$= p_s ds \cos\theta + wdxdy$$

Since the prism was of differential dimensions, the last term in the equation which represents the weight is then a differential of the second order and can be neglected. Substituting $\cos\theta = dx/ds$,

$$p_y dx = p_s ds \frac{dx}{ds} = p_s dx$$

or $$p_y = p_s$$

therefore $$p_x = p_y = p_s \tag{11}$$

This result would have been obtained regardless of the value of θ so that the intensity of pressure at a point in a fluid at rest is the same in all directions. This means that if the intensity of pressure at a point on a horizontal plane is known, the intensity of pressure at the same point on any other plane passing through the point is also known. The resultant action of this pressure on any plane area will be normal to it.

15. Intensity of Pressure Due to a Liquid Column. — As more and more substance is placed upon an area, the total force exerted on the area becomes greater and greater. If the size of the supporting area is maintained constant, the weight which each unit of this area must support becomes greater. In other words, the intensity of pressure at the base is a function of the height of the column.

Liquids at rest obey the same law. The greater the depth, the greater the intensity of pressure. Visualize a slender vertical column of liquid (Fig. 5) situated in a larger stationary body of the same liquid. Since the entire liquid mass is not moving, there can be no unbalanced forces acting. The forces acting on the vertical sides of the prism are horizontal. The forces in the vertical direction are the total pressure at the top and bottom of the prism and the weight of the liquid column. Let us now assume that our column has a unit cross-sectional area. The forces acting on the ends now become forces per unit area or simply the intensity of pressure. If the specific weight of the liquid is constant throughout the depth, the weight of the column is equal to the weight of a unit volume of the liquid times the volume or whA; but the area was assumed to be unity so the weight equals wh. Using the formula $\sum F_y = 0$ results in

$$p_2 = p_1 + wh \tag{12}$$

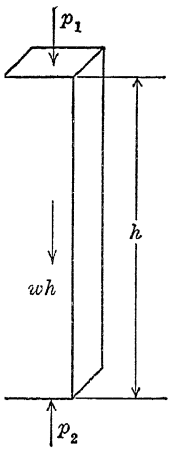


FIG. 5.

It must be understood that a consistent system of units must be used in Eq. (12). If p_1 and p_2 are in terms of pounds per square foot, w must be the specific weight in pounds per cubic foot and h the vertical height between the two points in feet. The student is advised to use these units. Equation (12) may also be written in the form

$$h = \frac{p_2}{w} - \frac{p_1}{w} \tag{13}$$

which is useful in determining the vertical distance between two points at which the pressures are p_1 and p_2.

Equations (12) and (13) apply only for liquids where the specific weight

may be considered constant. This condition is normally satisfied. The same equations could also be used for gases if the change in elevation were small.

When p_1 in the above equations is zero, as is generally the case at the surface of a liquid, then h is the vertical distance below the surface to the point where the intensity of pressure is p, or

$$h = \frac{p}{w}$$

The surface of a liquid at rest when in contact with the atmosphere is called a *free surface.* Except as affected by capillary attraction, the free surface of a liquid is level. The free surface is actually curved with every point equidistant from the center of the earth, but a limited portion of it may be considered a horizontal plane area. Since every point in a horizontal plane within the liquid is at the same distance from the free surface, the statement can be made that the intensity of pressure at the same elevation in a *connected* body of fluid is the same. Figure 6 illustrates what is meant by a connected body of fluid. The U-tube contains mercury in its lower portion with water above the mercury as shown. The mercury is a connected body of fluid, while the two water columns do not constitute such a body. Hence, the intensity of pressure at C and D are equal while those at A and B are not.

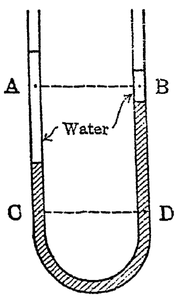


Fig. 6.

Any intensity of pressure may be considered as being produced by a column of liquid, the height of which depends upon the specific weight of the liquid chosen. For example, if the pressure at a point in a liquid is known to be 62.4 lb. per sq. ft., the height of a column of water which would cause the same intensity of pressure at its base would be,

$$h = \frac{p}{w} = \frac{62.4}{62.4} = 1 \text{ ft. of water}$$

If mercury with a specific gravity of 13.6 were used, the height of the equivalent mercury column would be,

$$h = \frac{p}{} = \frac{62.4}{13.6 \times 62.4} = 0.0734 \text{ ft.} \quad \text{or} \quad 12 \times 0.0734 \text{ in.}$$
$$= 0.881 \text{ in. of mercury}$$

Thus the intensity of pressure might have been given as 1 ft. of water or 0.881 in. of mercury, meaning that the intensity of pressure was equal to that which would be produced at the base of columns of fluids of this height.

The pressure itself may be caused by some entirely different means than by the column used to measure it.

Intensities of pressures are often given in pounds per square inch units. It is often found convenient to reduce these pressures to equivalent depths in terms of feet of water. One cu. ft. of water weighs 62.4 lb. and has an area at the base of 144 sq. in. The change in intensity of pressure for each foot depth of water is then $62.4/144 = 0.433$ lb. per sq. in. Using the inverse of this, there is a change of 1 lb. per sq. in. for each $1/0.433 = 2.308$ ft. change of depth. These are very convenient conversion factors and should be kept in mind. Conversion factors are easily forgotten, however, and they should not be used without forming a mental picture of how they were obtained.

PROBLEMS

15. Find the intensity of pressure at a point $\frac{1}{4}$ mile below the surface of a lake
(*a*) in terms of feet of water gage pressure;
(*b*) in terms of feet of water absolute pressure;
(*c*) in pounds per square inch gage pressure;
(*d*) in pounds per square inch absolute pressure.

16. Convert a pressure of 14.7 lb. per sq. in. to pressure in
(*a*) feet of water;
(*b*) inches of mercury, S.G. = 13.6;
(*c*) feet of oil, S.G. = 0.93;
(*d*) feet of air weighing 0.0807 lb. per cu. ft.

17. A tank is 30 ft. deep. The lower 20 ft. is filled with kerosene (S.G. = 0.8) and the upper part with air under a pressure of 25 lb. per sq. in. Find the intensity of pressure exerted upon the bottom of the tank in pounds per square inch.

18. A cubical box which is 1 ft. on a side has a pipe 1 in. in diameter extending 50 ft. above the top. Find the intensity of pressure at the bottom of the box when the box and the pipe are both filled with liquid having a specific gravity of 1.26.

Ans. $p = 4{,}010$ lb. per sq. ft.

19. For Prob. 18 find the intensity of pressure on the bottom of the box and the weight of the liquid removed if the liquid in the pipe is drained off.

16. Intensity of Pressure Due to a Gas Column. — The equations derived in Art. 15 for the variation of pressure in a liquid will not apply when the fluid is a gas because the specific weight, w, of the gas will vary significantly from point to point in the gas. For example, in Fig. 7 which represents a column of gas of unit cross-sectional area, the weight per unit volume at 2 would be less than the weight per unit volume at 1. If we consider a small element of the column dh high, however, w may be taken as constant for this element. If the intensity of pressure on the bottom face of this element is p, that on the top face will change slightly to $p + dp$. The only other force acting on the element will be its weight wdh.

Applying $\sum F = 0$, we obtain

$$p - (p + dp) - wdh = 0$$

or

$$dp = -wdh$$

The minus sign indicates that the pressure decreases as the height increases. In order that this equation may be integrated, it is necessary to know the law of variation of the specific weight, w. One such law for the case of gas is $pv = RT$ or $p/w = RT$ in which p is the absolute intensity of pressure, w the specific weight, T the absolute temperature, and R the gas constant. From this we obtain

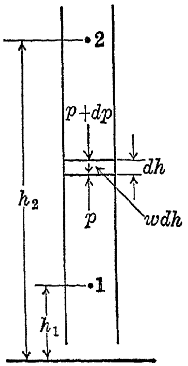


FIG. 7.

$$\frac{1}{w} = \frac{RT}{p}$$

therefore,

$$dh = -RT\frac{dp}{p}$$

If T is constant,

$$\int_{h_1}^{h_2} dh = -RT\int_{p_1}^{p_2} \frac{dp}{p}$$

Integrating,

$$h_2 - h_1 = -RT \log_e \frac{p_2}{p_1} = RT \log_e \frac{p_1}{p_2} \tag{14}$$

Equation (14) assumes practical importance in the determination of the pressure variation in the atmosphere. The temperature of the atmosphere decreases at the rate of 1° F. per 300 ft. increase in elevation up to an elevation of about seven miles. Above this the temperature remains constant. The rate of change in temperature is so small that Eq. (14) can be used without serious error even for large differences in elevation by taking T as the mean of the absolute temperature at the lower and upper point. For example, assuming a ground temperature of 80° F., and pressure of 14.7 lb. per sq. in. abs., the pressure at an elevation of 36,000 ft. computed in one step is 3.60 lb. per sq. in. abs. The pressure at the same elevation computed by applying the equation to points 3000 ft. apart and adding the changes in pressures for the twelve 3000 ft. intervals gives a pressure of 3.58 lb. per sq. in. abs. The actual measured pressure at this elevation, as given by Humphreys, is 3.25 lb. per sq. in. The error between measured and computed values is about 0.35 lb. per sq. in. in a total difference in pressures over the 36,000 ft. of about 11 lb. per sq. in. This represents an error of about 3 per cent in the computed pressure difference.

PROBLEMS

20. A gas in a main at a point where the pressure is 16.0 lb. per sq. in. abs. has a specific weight of 0.051 lb. per cu. ft. At a point in a main 1600 ft. higher, the specific weight of the gas is 0.047 lb. per cu. ft. Assuming the specific weight to vary uniformly between these two points, find the absolute pressure at the higher elevation.

Ans. p = 15.46 lb. per sq. in. abs.

21. The atmospheric pressure at an elevation of 500 ft. above sea level is 29.67 in. of mercury. If the temperature is 60° F. at this elevation and decreases 1° F. per 300 ft., find the pressure at an elevation of 12,500 ft. above sea level. Take $R = 53.3$.

17. The Barometer. — The atmosphere surrounds the earth and extends upward for many miles. The pressure which it exerts on the earth at any one locality changes from time to time. Experience has taught us that certain changes of weather occur with certain changes of atmospheric pressure. In addition to the local change in pressure there is a change due to an increase in elevation above sea level. In the preceding article, it was shown that if the pressures at two points in the atmosphere were known, it would be possible to get a fairly good approximation as to the difference in elevation between the two points, Eq. (14). It therefore becomes necessary to have an instrument which is capable of measuring the atmospheric pressure.

A barometer is an instrument for indicating atmospheric pressure. A common type is shown in Fig. 8. It consists of a glass tube which is closed at its upper end. The tube is filled with mercury after which the open end is placed in a dish of mercury. The tube must be sufficiently long so that the mercury will not extend all the way to the top. The portion which is not filled contains a small amount of mercury vapor which exerts a slight pressure on the top of the mercury column in the tube. This small vapor pressure will be neglected in this discussion.

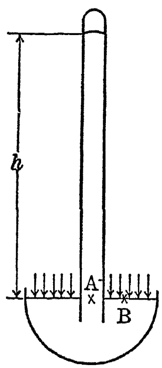


Fig. 8. A simple barometer.

For equilibrium, the pressure of the atmosphere acting on the surface of the mercury in the dish must be balanced by the pressure at A. The atmospheric pressure at B is equal to the pressure in the mercury at A, but the pressure at A is the pressure produced by a column equal to h in height. Thus h is a measure of the atmospheric pressure and is usually measured in inches of mercury. A scale is ordinarily attached to the side of the tube to facilitate the taking of readings.

This type of barometer is cumbersome if it is to be transported over large distances such as would be required in determinations of differences

in elevations. For this purpose more compact ones are manufactured which depend upon the action of the atmosphere on a flexible diaphragm which forms one side of a partially evacuated metal box. The motion of the diaphragm actuates a needle and the instrument may be calibrated so that the needle reads directly approximate elevations above sea level.

18. Consideration of Atmospheric Pressure. — In the majority of problems dealing with liquids, it is a difference between the intensities of two pressures that is required. Since the pressure of the atmosphere, p_a, contributes the same amount to each of these pressures, it may be neglected in getting the difference. Consider the side wall of a vessel containing a liquid, for example, Fig. 9. In finding the total pressure exerted by the liquid on the side wall it will be found necessary to obtain the resultant intensity of pressure of the fluid on the wall at points such as O. If p_a is the atmospheric pressure, the pressure on the inside face at O including atmospheric pressure is $p_a + wh$; that on the outside is p_a. Their resultant is $(p_a + wh) - p_a$, or wh. Thus it is necessary to consider only those pressures caused by forces other than the pressure of the atmosphere.

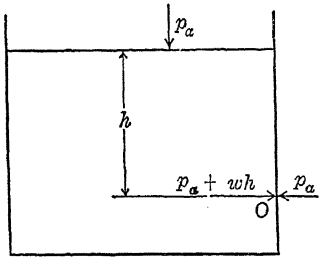


FIG. 9.

Attention is again called to the fact that while atmospheric pressure should not usually be considered in the solution of problems involving liquids, it must always be considered where a gas is involved.

19. Measurement of Pressure. — The instruments used most often for the measurement of pressure are the piezometer tube, the simple U-tube, and the Bourdon gage. A brief description of each of these follows.

The Piezometer Tube. This is the simplest of all the devices for the measurement of the intensity of pressure at a point in a liquid. It consists merely of a tube attached to the vessel in which the pressure is required, as shown in Fig. 10, and utilizes a column of liquid of the same kind as that in the vessel for measuring the pressure. The pressure at A, for example, would be equal to h feet of the liquid. This height can be expressed in terms of another liquid or can be converted into a pressure in force per unit area if the specific weight of the liquid is known.

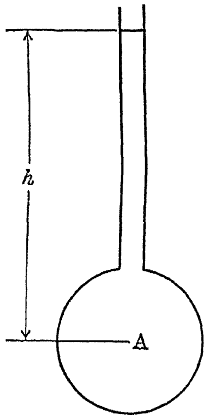


FIG. 10.

The connection of the tube to the vessel shown can be made at any point of the circumference because the liquid will rise to the same elevation regardless of the point of connection. The pressure will always be equal to the height of the column of liquid in the tube above the point at which the pressure intensity is desired.

In order to avoid the effect of capillary action in the measurement of h, small tubes should not be used. The capillary action will be negligible in tubes which are $\frac{1}{2}$ in. or more in diameter.

When large pressures are to be measured, the length of the piezometer tube becomes too great and other methods must be used.

The Simple U-Tube. The simple U-tube is used for measuring small pressures such as might exist in a vessel containing a gas, or pressures larger than those that could be measured easily with the piezometer tube. The only difference in the application of the U-tube to the measurement of small or large pressures is in the liquid used as a measuring fluid. For low gas pressures water is usually employed and for higher pressures mercury is used because of its greater weight. Figure 11*a* shows a U-tube containing mercury attached to a vessel within which the intensity of pressure is required. The greater pressure at D causes the mercury to rise above D

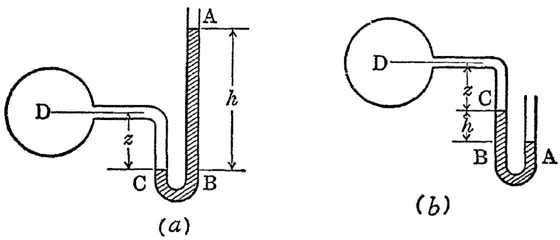


FIG. 11. Manometers for reading (*a*) positive pressures and (*b*) negative pressures.

in the U-tube. Remembering that the pressure in a fluid increases with the depth, the pressure at A is atmospheric; that at B and C is greater than atmospheric by h feet of mercury; and that at D is less than that at C by z feet of the fluid in the vessel. The h feet of mercury is equivalent to a higher column of the liquid in the vessel and may be converted to a height in terms of that liquid if the specific gravities of the two liquids are known. For example, if the Specific Gravity of mercury is 13.6 and that of the liquid in the vessel is 1.36, the weight of mercury is $13.6/1.36 = 10$ times as great. A column of mercury 1 ft. high will have the same pressure at its base as a column 10 ft. high of the liquid in the vessel. Thus the h ft. of mercury is equivalent to $10h$ ft. of liquid. The pressure at D, there-

fore, in terms of feet of fluid in the vessel would be

$$\frac{p_D}{w} = 10h - z \text{ above atmospheric}$$

The student should employ the line of reasoning illustrated above in solving problems in which fluid columns are used in measuring pressures.

If the fluid in the vessel were a gas, the difference in pressures between C and D would be very small so that the pressure could be taken equal to that at C, or h ft. of mercury.

Figure 11b shows a U-tube used to measure a pressure which is less than atmospheric, or a partial vacuum. The pressure at A and B is atmospheric; that at C is less than atmospheric by h ft. of mercury; that at D is less than that at C by z ft. of liquid in the vessel. Using the same ratio of the specific gravities as in the previous illustration, the pressure at D would be

$$\frac{p_D}{w} = 10h + z \text{ ft. of liquid less than atmospheric}$$

If atmospheric pressure is taken as zero this pressure would be

$$\frac{p_D}{w} = -(10h + z) \text{ ft. of liquid}$$

The Bourdon Gage. Bourdon gages are usually used for measuring high pressures, such as those in a steam boiler or high pressure water main.

The essential element of a Bourdon gage is a curved, flattened tube made of non-corroding metal. One end of the tube is open, and fixed rigidly to some support; the other end is closed, and free to move. When fluid under pressure is admitted into the curved tube at the fixed end, the internal pressure tends to straighten out this tube. The free end of the tube is connected to an indicating needle by means of levers, a rack, and a pinion. Any movement of the free end of the tube causes the needle to move, and to indicate a pressure on a properly calibrated dial.

Bourdon gages indicate pressures referred to atmospheric pressure as zero. Some gages are so constructed that they can read pressures less than atmospheric as well as those greater. Gages of this type are called vacuum gages, and have dials graduated in both the positive and negative directions from zero. A negative reading on such a gage would mean a pressure less than atmospheric by the amount indicated by the dial.

The Bourdon gage gives the pressure at a point in the fluid where the elevation is equal to that of the center of the gage. If the pressure at some other point in the fluid is required, proper correction must be made for the difference in elevation between the point and the center of the gage. Ordinarily, this correction is not necessary when gas pressures are being measured.

PROBLEMS

22. Pipe A of Fig. 12 carries carbon tetrachloride (S.G. = 1.594). The manometer contains mercury and is open to the atmosphere. Find the intensity of pressure at the center of the pipe.

Ans. $p_A = -7.26$ lb. per sq. in.

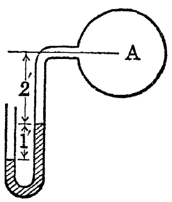


FIG. 12.

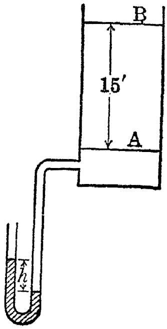


FIG. 13.

23. A mercury U-tube is attached to a tank containing water as shown in Fig. 13. When the surface of the water in the tank is at A, the value of h is 2.2 in. Find the value of h when the surface of the water in the tank is raised to B, 15 ft. above A.

24. A Bourdon gage attached to a tank containing air reads a vacuum of 8 in. of mercury. What is the absolute pressure in the tank in inches of mercury if the atmospheric pressure is 14.7 lb. per sq. in?

20. Measurement of Difference in Pressures. — In many problems involving the flow of fluids, it is necessary to know the difference in pressures existing at two points in the fluid situated some distance apart along the pipe or conduit through which the fluid is flowing without knowing the value of either one of the pressures. The method for determining this difference in pressure is exactly the same as that employed in finding the difference in pressures existing in two closed vessels except that certain precautions must be observed in making connections of pressure measuring devices to a pipe containing fluids in motion that are not necessary when the fluids are at rest. These precautions will be pointed out in a later chapter. The discussion here will be limited to the determination of the difference in pressures existing in vessels containing fluids at rest.

The device used for measuring differences in pressure is called the *differential gage* or *manometer*. It consists of a U-tube which contains a fluid either heavier or lighter than the fluids whose pressure difference is desired. The choice of gage fluid will depend upon the magnitude of the difference of pressures which one desires to measure.

A simple differential gage is shown in Fig. 14. The U-tube is connected to the two sources of pressure of which the difference is required. The bottom of the U-tube contains some liquid which has a specific gravity greater than that of the fluids in the vessels to which the tube is connected. In certain cases, it is convenient to use a gage fluid with a specific gravity less than that of the two fluids of which the pressure difference is being measured. In that case the U-tube is inverted.

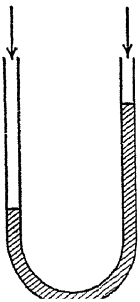


FIG. 14.

The principle of the differential gage is based upon balancing columns of fluids of different specific gravities. It is convenient to reduce all pressures and fluid columns to equivalent columns of a single liquid, usually water. The required difference in pressure will then be expressed in terms of a height of a column of this fluid, and may be converted into a difference in any other units desired. Several illustrations of the use of the differential gage follow.

In Fig. 15 are shown two vessels, A and B, containing water under pressure. A mercury U-tube is connected to the two vessels and shows a displacement, h; z_1 and z_2 are vertical distances as shown and can be measured. Calling the pressures at A and B, p_a and p_b in pounds per square foot respectively, these pressures in feet of water would be p_a/w and p_b/w in which w is the specific weight of water. The difference $p_a/w - p_b/w$ is the required quantity.

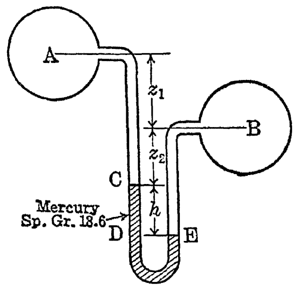


FIG. 15.

Starting at A, one works around towards B, remembering that, as one goes toward a greater depth, the pressure increases, and that at points having the same elevation in a connected body of fluid, the pressures are equal. The pressure at C is greater than that at A by $(z_1 + z_2)$ ft. of water; that at D is greater than that at C by h ft. of mercury or by the specific gravity of mercury times h ft. of water. The pressure at E equals that at D, and the pressure at B is less than that at E by $(h + z_2)$ ft. of water. Writing this statement in the form of an equation, we have

$$\frac{p_a}{w} + (z_1 + z_2) + (\text{S.G.})\, h - (h + z_2) = \frac{p_b}{w}$$

collecting,

$$\frac{p_b}{w} - \frac{p_a}{w} = (\text{S.G.} - 1)\, h + z_1 \qquad (13.6 - 1)\, h + z_1$$

This equation indicates that the pressure at A is less than that at B by (12.6) $h + z_1$ ft. of water. The student should realize that the equation applies only for this problem and is given only to illustrate the procedure involved in solving a differential gage problem. Any one of the quantities in the above problem could have been determined by this method if the others were known.

The student will notice that the coefficient of h in the above equation depends upon the specific gravity of the measuring liquid. If the specific gravity is small, the value of h required for a given difference in pressures is large, so that it would be possible to obtain a value of h larger than the differences in pressures in feet of liquid. A manometer using a measuring fluid with a specific gravity near 1, would thus magnify the difference in pressures; on the other hand, if the specific gravity of the fluid is large the reading, h will be much smaller than the difference in pressure heads to be determined. Measuring fluids with small specific gravities would therefore be used for small differences in pressures, while fluids such as mercury, with high specific gravities, would be used for large differences.

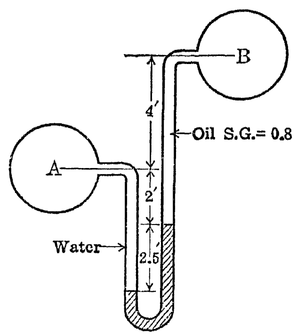


FIG. 16.

Another example will now be considered in which A contains water under a pressure of 10 lb. per sq. in. and B contains oil which has a specific gravity of 0.8 and has an unknown pressure. The dimensions are as shown in Fig. 16.

$$p_a = 10 \text{ lb. per sq. in.,} \quad \text{so} \quad \frac{p_a}{w} = 23.08 \text{ ft. of water}$$

One then proceeds in the same manner as before.

$$23.08 + 4.5 - 13.6\,(2.5) - 0.8\,(6) = \frac{p_b}{w}$$

so
$$\frac{p_b}{w} = 23.08 + 4.5 - 34.0 - 4.8 = -11.22 \text{ ft. of water}$$

This is a partial vacuum of 11.22 ft. of water, or 4.86 lb. per sq. in.

The same principle always applies, namely: start from the point of known condition and from that proceed toward the point of unknown condition. One must always keep in mind that the pressures and changes in pressures due to difference in elevations should be expressed in the same units. Any

desired conversion of units may be made after the problem has been solved. It is not possible to carry two types of units along in the solution of a problem.

PROBLEMS

25. Pipe *A* of Fig. 17 contains water under a pressure of 5 lb. per sq. in., and pipe *B* contains a liquid having a specific gravity of 1.05. Find the intensity of pressure at the center of pipe *B*.

26. Pipe *A* of Fig. 18 contains water under a pressure of 10 lb. per sq. in., and pipe *B* contains oil having a specific gravity of 0.8. Pressure in pipe *B* is 15 lb. per sq. in. The measuring fluid is mercury. Find the height *h*.

Ans. $h = 0.387$ ft.

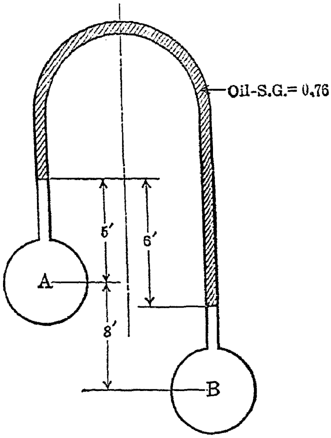


Fig. 17.

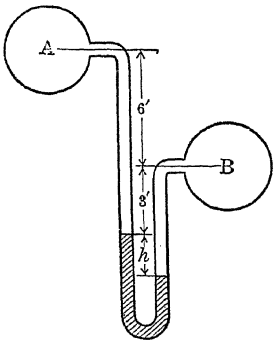


Fig. 18.

27. Pipe *A* of Fig. 19 contains oil which has a specific gravity of 0.9 and pipe *B* contains water. Find the intensity of pressure in terms of feet of water at points *A*, *B*, *C*, and *D*.

21. Multiplying Gages. — A number of devices have been developed for measuring small differences of pressure. These differences might be between two gases, or two liquids. Two such devices will be described.

Inclined U-tube. The inclined U-tube shown in Fig. 20 is capable of measuring differences of either gaseous or liquid pressures. The manometer reading is given by

$$h = R \tan \theta \tag{15}$$

where R has been obtained from a horizontal scale. In case the gage has been used for obtaining the difference in gaseous pressures, water would be used for a measuring fluid and h gives the difference in pressure directly

in inches of water. Should the manometer be used for measuring differences in fluid pressures, h would be interpreted as outlined in Art. 20.

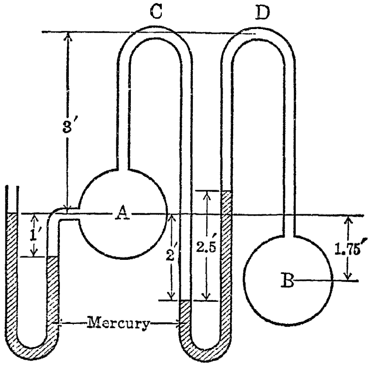


FIG. 19.

It is advisable that gages of this type be calibrated due to possible differences in the diameters of the tubes and in the angles of inclination of the tubes. The differences in diameters would introduce errors due to a different capillary rise in the two tubes.

The draft gage which is used in power plants operates on the same principle as this gage. Gages of this type can be safely used for a magnification of ten times.

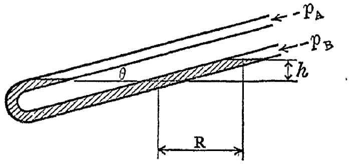


FIG. 20.

Two-fluid U-tubes. The two-fluid U-tube shown in Fig. 21 is very sensitive for measuring small gas heads. The two reservoirs are made large in order to make a compact gage and in order to eliminate the effect of capillarity. Let A be the cross-sectional area of each of the reservoirs and a that of the tube forming the U; let s_1 be the specific gravity of the lighter fluid and s_2 that of the heavier fluid. Then the difference in pressure in feet of water is given by the expression

$$\frac{p_A}{w} - \frac{p_B}{w} = (h)\left(s_2 - s_1 + \frac{a}{A}s_1\right) \tag{16}$$

where the reading h is in feet.

When a/A is sufficiently small, the last term in the above expression becomes negligible in comparison to the difference $(s_2 - s_1)$, but it should not be neglected without proper consideration. Since the magnification of this gage becomes great as the difference $(s_2 - s_1)$ becomes small, two fluids should be chosen for which that difference is very small.

Where a high degree of precision is desired, the gage should be calibrated at the working temperature. This gage can be used for a magnification of 25 to 30 times.

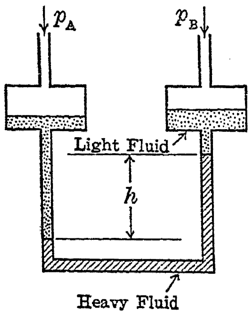


FIG. 21.

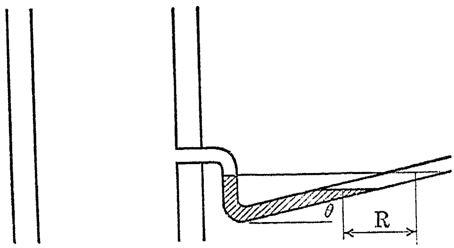


FIG. 22.

PROBLEMS

28. The inclined U-tube containing water, Fig. 22, is used to measure the draft in a chimney. If the reading, R, is 10.4 in. and θ is 11° 18′ what is the pressure in the chimney, referred to atmospheric, in inches of water?

29. Derive Eq. (16) for the two-fluid manometer shown in Fig. 21. Note: when the pressures, p_A and p_B, are equal, the fluids in the reservoirs are at the same elevation and h is zero.

30. A two-fluid manometer of the type shown in Fig. 21 is used to measure the difference between the pressures of two vessels containing gases. The specific gravity of the lighter fluid in the manometer is 0.842 and that of the heavier 0.916. The ratio of the cross-sectional area of the reservoirs to that of the U-tube is 10. Find the difference in pressures when h is 1.35 in.

CHAPTER III

FORCE OF FLUID PRESSURE ON AREAS

22. Introduction. — The engineer is often called upon to design structures which must withstand forces caused by fluid pressure. Gates, dams, valves, pipelines, and tanks are only a few of the many structures requiring a knowledge of the force of fluid pressure for their design.

The forces to be considered are surface forces distributed over plane or curved areas, and act everywhere in a direction perpendicular to the area. In general, the intensity of pressure on any element of area varies from point to point in the area. If we choose a small element of area, dA, at which the intensity of pressure is p, the force acting on this element will be pdA.

In many problems in mechanics, the effect of a force system on a body is best studied by finding the resultant of the system, or by finding the components of the resultant in two perpendicular directions. If the system consists of surface forces such as those caused by fluid pressure, the resultant is the resultant of a group of variable forces each equal to pdA acting normal to the area dA. The magnitude, direction, and line of action of this resultant must be found.

23. Magnitude and Direction of Fluid Pressure on Plane Areas. — Figure 23 shows two views of an irregularly shaped area situated in a plane

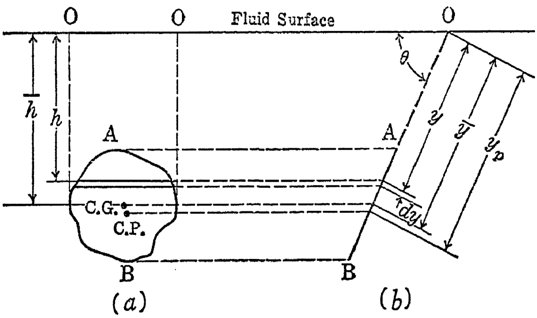


FIG. 23

making an angle θ with the horizontal fluid surface. Figure 23a is the area as it would appear if viewed from a position parallel to the fluid surface, while Fig. 23b is a side view of the plane of the area. Let it be

required to find the resultant of the fluid pressures acting on one side of the area in magnitude and direction.

The area may be divided into differential elements, each of an area dA, located at an inclined distance, y, from the surface. The vertical distance to the elementary area is $h = y \sin \theta$. The intensity of pressure, p, at the depth, h, is wh, and the force acting on dA is equal to the intensity of pressure times the area, or

$$dF = pdA = whdA = wy \sin \theta dA$$

Each of the forces such as dF is normal to the area and their resultant will be a force, normal to the area, whose magnitude is the sum of the elementary forces. Integrating the expression given above for dF, we have

$$F = \int dF = \int wy \sin \theta dA = w \sin \theta \int ydA$$

But $\int ydA$ is the moment of the area about the line in the surface which is the intersection of the plane of the area and the fluid surface. Thus

$$\int ydA = A\bar{y}$$

where $\bar{y}$ is the inclined distance from the centroid of the area to the surface and A is the whole area. Therefore,

$$F = w\bar{y} \sin \theta A$$

But $\bar{y} \sin \theta$ is equal to $\bar{h}$, the vertical distance from the surface to the centroid of the area, so that

$$F = w\bar{h}A \tag{17}$$

Thus, the resultant force on one side of a plane surface is perpendicular to the plane, and equals the area times the intensity of pressure at its centroid. The student will note that the magnitude of the resultant force for a given area will remain the same, regardless of the angle θ, as long as the centroid of the area remains at the same depth below the surface.

The resultant force, F, will be expressed in pounds when w is the weight of the fluid in pounds per cubic foot; $\bar{h}$ is in feet; and A is in square feet.

24. Center of Pressure — Position of Resultant Force on Plane Areas. — The point at which the resultant of the fluid pressures intersects the area is called the *center of pressure*. Its location is determined in the case of plane areas by using the principle of moments which states that the moment of the resultant of a system of forces is equal to the moment of the separate forces about the same axis. Referring to Fig. 23 and designating the inclined distance to the center of pressure by y_p, the moment of

the resultant, F, about 0 is Fy_p. The moment of each differential force is ydF and the total moment of these forces about 0 is $\int ydF$. Applying the principle of moments, we have

$$Fy_p = \int ydF$$

Substituting the values of F and dF from the preceding article, it follows that

$$wA\bar{y} \sin \theta y_p = \int yw \sin \theta y dA$$

$$= w \sin \theta \int y^2 dA$$

$$y_p = \frac{\int y^2 dA}{A\bar{y}} = \frac{I}{A\bar{y}} \tag{18}$$

In this equation I is the moment of inertia of the area about a horizontal axis which lies in the plane of the area at the surface of the fluid. Ordinarily it is the moment of inertia about a centroidal axis that is known so that it is convenient to replace I by $I_0 + A\bar{y}^2$, where I_0 is the moment of inertia of the area about a parallel centroidal axis and $\bar{y}$ is the distance between the centroidal axis and the horizontal axis at the surface. Then

$$y_p = \frac{I_0 + A\bar{y}^2}{A\bar{y}} = \bar{y} + \frac{I_0}{A\bar{y}} \tag{19}$$

Since

$$\frac{I_0}{A} = k^2$$

where k is the radius of gyration of the area with respect to a centroidal axis, Eq. (19) may be written as

$$y_p = \bar{y} + \frac{k^2}{\bar{y}} \tag{20}$$

The student should bear in mind that y_p is an *inclined* distance to the point where the resultant of the fluid pressure acts. It is important to note, also, that *the resultant acts at a point which is always below the centroid of the area in question by a distance $k^2/\bar{y}$, measured in the plane of the area.* The value of $\bar{y}$ may be increased either by placing the area at a greater depth or by rotating the area about its centroidal axis so that it becomes more nearly parallel to the surface. If $\bar{y}$ is increased indefinitely by either

of the methods mentioned above, $k^2/\bar{y}$ approaches zero and in the limiting case when $\bar{y}$ is infinite, the center of pressure coincides with the centroid of the area. This is the condition when the area lies in a horizontal plane.

Practically all areas dealt with in problems of this kind are symmetrical with respect to a vertical centroidal plane perpendicular to the area. For all such areas, the center of pressure is located on the intersection of this plane of symmetry and the area.

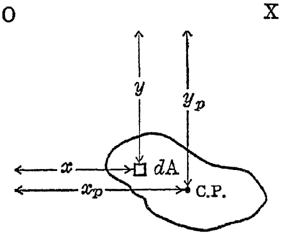


FIG. 24.

In cases where the area does not have an axis of symmetry of the kind described above, the lateral position of the center of pressure may be found by taking moments of the elementary dF forces about another axis, at right angles to the one used in determining y_p, such as Y in Fig. 24.

The force acting on dA is

$$dF = wy \sin \theta dA$$

Taking moments about the Y axis and calling x_p the distance from Y to the center of pressure, we obtain

$$F \cdot x_p = \int xwy \sin \theta dA$$

But

$$F = w \sin \theta A\bar{y}$$

Therefore

$$w \sin \theta A\bar{y}x_p = w \sin \theta \int xydA$$

or

$$x_p = \frac{\int xydA}{A\bar{y}} \tag{21}$$

Equation (20) is similar to Eq. (18) except that the expression, $\int xydA$, appears instead of $\int y^2dA$. The $\int xydA$ is known as the product of inertia about the X-Y axes and is sometimes designated by the symbol P_{xy}. Its value can be found for the common areas by direct integration with respect to the X-Y axes. Most of the handbooks give values of the product of inertia about a pair of centroidal axes, and if its value is desired about a pair of parallel axes such as X-Y in this derivation, a transfer equation may be used. This equation is

$$P_{xy} = P_{\bar{x}\bar{y}} + A\bar{x}\bar{y} \tag{22}$$

where P_{xy} is the product of inertia with respect to the X-Y axes, $P_{\bar{x}\bar{y}}$ is the

product of inertia with respect to a pair of parallel centroidal axes, and $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of the area referred to the X-Y axes.

PROBLEMS

31. A rectangular gate which is 6 ft. wide and 4 ft. high lies in a vertical plane. The water surface coincides with the top edge of the gate. (*a*) Find the force exerted by the water upon the gate. (*b*) Find the position of the center of pressure.

32. If the gate of Prob. 31 has the top edge 10 ft. below the surface of the water, (*a*) find the force exerted by the water upon the gate, (*b*) find the position of the center of pressure.

Ans. (*a*) $F = 17{,}970$ lb., (*b*) $y_p = 12.11$ ft.

33. A triangular gate which has a base of 4 ft. and an altitude of 6 ft. lies in a vertical plane. The vertex of the gate is 2 ft. below the surface of a tank which contains oil having a specific gravity of 0.86. (*a*) Find the force exerted by the oil upon the gate. (*b*) Find the position of the center of pressure.

34. A gate which has a 4 ft. base and a 6 ft. altitude is in the shape of a right triangle. The base is horizontal and the gate lies in a vertical plane with the vertex 2 ft. below the water surface. Find the horizontal and vertical positions of the center of pressure. Let x_p be measured from the vertical side of the gate.

Ans. $x_p = 1.44$ ft.

35. A pipe line which is 8 ft. in diameter contains a gate valve. The pipe contains water and the pressure at the center of the pipe is 12 lb. per sq. in. (*a*) Find the force exerted by the water upon the gate. (*b*) Find the position of the center of pressure.

Ans. (*a*) $F = 86{,}800$ lb., (*b*) 0.144 ft. below center.

36. A rectangular gate is 8 ft. wide and 6 ft. high. The plane of the gate makes an angle of 60° with the horizontal and the top of the gate is 5 ft. below the surface of the water. (*a*) Find the force exerted by the water upon the gate. (*b*) Find the position of the center of pressure.

Ans. (*a*) $F = 22{,}800$ lb., (*b*) $y_p = 9.11$ ft.

37. A rectangular gate is in a vertical bulkhead. The gate is 12 ft. wide and 8 ft. high. The water surface on the upstream side is 10 ft. above the top of the gate and that on the downstream side is 2 ft. above the top of the gate. (*a*) Find the force exerted by the water on the upstream side of the gate. (*b*) Find the force exerted by the water on the downstream side of the gate. (*c*) Find the net force acting upon the gate and the position at which it acts.

38. Given the same conditions as in Prob. 37 except that the water on the downstream side of the gate is 3 ft. below the top of the gate. (*a*) Find the force exerted by the water on the upstream side of the gate. (*b*) Find the force exerted by the water on the downstream side of the gate. (*c*) Find the net force acting upon the gate and the position at which it acts.

Ans. (*c*) $R = 74{,}500$ lb. acting 0.136 ft. below center.

39. A gate which is 6 ft. wide and 4 ft. high lies in a vertical plane and is hinged at the bottom. There is a substance on the upstream side of the gate which extends 5 ft. above the top of the gate and which has a specific gravity of 1.45. There is air on the downstream side of the gate. (*a*) Find the force which is exerted upon the gate. (*b*) Find the position of the center of pressure. (*c*) Find the least force acting horizontally at the top of the gate which is capable of opening it.

Ans. (*c*) $F = 6880$ lb.

40. Find the least force acting horizontally at the top of the gate of Prob. 39 which is capable of opening it if in addition, there is water up to the top of the gate on the downstream side.

25. Fluid Pressure on Curved Areas. — In engineering problems, it is often convenient to deal with the horizontal and vertical components of a force rather than with the force itself. This is true when the force exerted by the fluid is against a curved surface. Let us consider the action of a fluid against the curved face of the dam in Fig. 25*a*. This problem may be attacked in one of two ways: either the force exerted upon a differential area may be considered and then the resulting expression can be integrated to give the total force, or the liquid block *abc* may be considered to be in equilibrium and the forces acting upon this block may be found. In general, the equation of the curved surface cannot readily be written. This prevents the first method from lending itself to an easy method of solution.

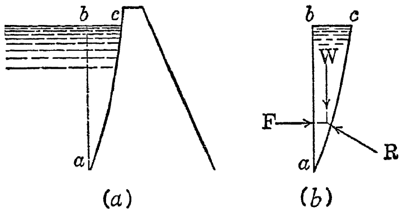


Fig. 25.

A free-body diagram of the block of water (*abc*) is shown in Fig. 25*b*. In this diagram, it is considered that *bc* is a free surface upon which only atmospheric forces are acting.

Considering the equation $\sum F_x = 0$, it is evident that F must equal the horizontal component of the force R which the face of the dam exerts upon the water block. The force which the water exerts upon the dam is equal in magnitude to the force R, but acts in the opposite direction. In order to determine the magnitude of the horizontal component of the force exerted by the water upon the face of the dam, it is then only necessary to apply Eq. (17). The position of this component will be that determined by the use of Eq. (20).

Likewise, considering the equation $\sum F_y = 0$, it is evident that the vertical component of the force R must be equal to the weight W of the water block. The weight W acts at the centroid of the area *abc*. The combination of these two forces (F and W) according to the principles of mechanics would give the position at which the actual force would intersect the face of the dam. As stated before, it would seldom be necessary to locate this point.

It is possible that the surface *bc* would not be a free surface subjected only to atmospheric pressure. In this case, the value of F would be determined by means of Eq. (17) in which $\bar{h}$ would have a value equal to the depth of equivalent fluid in order that the pressure at the centroid of the

projected area ab would be the same as it actually is with the added pressure. The value of W would be replaced by a value which would be the sum of the weight of the water block and of the force which would be exerted by the pressure upon the surface, $(W + F_y)$ of Fig. 26. The position of the resulting vertical force would be found by using the equation of mechanics, namely

$$(W + F_y)x = Wx_1 + F_yx_2$$

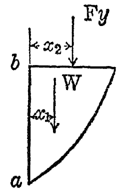


FIG. 26.

Illustrative Problem: Find the position at which the resultant pressure will cut the base of the dam shown in Fig. 27*a*. The free-body diagram of the dam is shown in Fig. 27*b*.

It is first necessary to find the magnitude and position of the different forces which are acting upon the dam. The area of the water block is equal to 76.3 sq. ft. and has a centroid which is 2.83 ft. from the upstream side. The area indicated

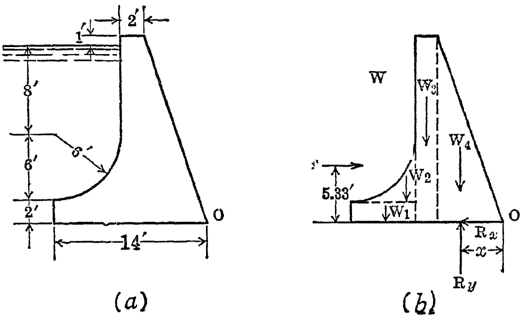


FIG. 27.

by W_2 is equal to 7.7 sq. ft. and its centroid is 1.32 ft. from its downstream edge. The areas and positions of the centroids of the other areas are evident. One foot length of dam will be considered and the concrete will be considered to weigh 150 lb. per cu. ft. With the above in mind, it follows that

$$\begin{aligned} W &= (76.3)(62.4) = 4765 \text{ lb.} \\ W_1 &= (12)(150) = 1800 \text{ lb.} \\ W_2 &= (7.7)(150) = 1155 \text{ lb.} \\ W_3 &= (34)(150) = 5100 \text{ lb.} \\ W_4 &= (51)(150) = 7650 \text{ lb.} \\ F &= 62.4\,(8)(16) = 7985 \text{ lb.} \\ R_y &= 20470 \text{ lb.} \end{aligned}$$

Taking moments with respect to point 0, we obtain $20470x = 4765(11.17) + 1800(11) + 1155(9.32) + 5100(7) + 7650(4) - 7985(5.33)$

$$x = \frac{107530}{20470} \quad 5.25 \text{ ft.}$$

26. Consideration of the Safety of a Dam. — Article 25 illustrated the method for determining the horizontal and vertical components of the resultant of a system of forces acting on a dam. In practice, the difficulty is not in the determination of the resultant of a known set of forces but in the determination of the magnitude and position of some of the forces themselves. For example, little is known about the magnitude of ice forces or wind forces to which a dam may be subjected. In addition, there is always the possibility that water will have access to the base of the dam and thus cause an upward pressure of unknown magnitude. A discussion of such forces is beyond the scope of this text and the interested student should consult standard works on the design of dams.

The resultant of the forces mentioned above is held in equilibrium by the reaction of the foundation on the base of the dam. The foundation reaction is a distributed force which must be capable of balancing the horizontal and vertical components of the resultant of the other forces. It is usually divided into a variable pressure normal to the base and a tangential force parallel to the base.

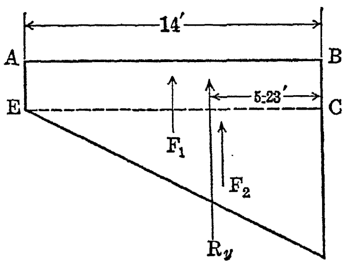


FIG. 28.

Two methods for determining the maximum and minimum unit normal stresses at the base of any section of a dam are in common usage. Both of these methods assume that the pressure diagram varies as a straight line. The two methods will now be demonstrated for the illustrative problem of Art. 25. One of these methods will use the mechanics equation

$$F\bar{y} = F_1\bar{y}_1 + F_2\bar{y}_2 + \cdots$$

while the other will use the strength of materials equation

$$S = \frac{P}{A} \pm \frac{Mc}{I} \tag{24}$$

Let $ABDE$ of Fig. 28 represent the pressure diagram at the base of the dam. AB is either the width of the dam, or the area of the base of the dam per foot of length of dam. We will here consider AB as an area. Then AE and BD are the unit stresses at the base of the dam expressed in pounds per square foot.

From $\sum F_y = 0$, we obtain

$$F_1 + F_2 = R_y \tag{a}$$

From $\sum M_B = 0$, we obtain

$$7F_1 + 4.67F_2 = 5.23R_y \tag{b}$$

Solving (a) and (b) we obtain,

$$F_1 = 4951 \text{ lb.}$$

$$F_2 = 15519 \text{ lb.}$$

$$AE = \frac{4951}{14} = 354 \text{ lb. per sq. ft.}$$

$$CD = \frac{15519 \times 2}{14} = 2217 \text{ lb. per sq. ft.}$$

The unit compressive stress at the heel is then 354 lb. per sq. ft. and that at the toe is $BC + CD = 2571$ lb. per sq. ft.

These same problems will now be solved by the second method. In this equation the resultant force R_y is the force P. The area, A, is the area per foot of length of dam, or 14 sq. ft. The moment is obtained by taking the product of R_y by its distance from the center of the base. The value c is the distance from the center of the dam to the heel, or the toe. The moment of inertia is written for a rectangular area which is 1 ft. wide and of a height equal to the distance from the heel to the toe of the dam; thus,

$$I = \frac{1 \times 14^3}{12} = 228.7 \text{ ft.}^4$$

These values are then substituted in Eq. (24) to give

$$S = \frac{20470}{14} \pm \frac{20470\ (7 - 5.23)(7)}{228.7}$$

$$= 1462 \pm 1109$$

From which $\qquad AE = 353$ lb. per sq. ft.

and $\qquad BD = 2571$ lb. per sq. ft.

These unit stresses must not exceed the safe values for the material of the dam, or of the foundation.

A study of Eq. (24) would show that if the resultant cut the base at a distance greater than one-sixth the base downstream from the centroid of the base, the last term of the equation would be greater than the second and tension would occur at the upstream edge. Since it is never assumed that masonry can resist tensile forces, dams are not considered safe unless the resultant cuts the base within the middle third.

In order for the dam to be considered safe against sliding, the ratio of R_x and R_y must not exceed some predetermined value. The magnitude of this value will depend upon the nature of the foundation materials. For the purposes of this text, this ratio will be considered satisfactory whenever its magnitude does not exceed 0.6. For the dam under consideration,

$$\frac{R_x}{R_y} = \frac{7985}{20470} = 0.39$$

which is a satisfactory value.

To summarize, the conditions for safety of a dam or any portion of a dam above a given section are:

(1) The resultant should cut the base within the middle third in order to eliminate tensile stresses.

(2) The allowable pressures should not be exceeded.

(3) The horizontal force developed by the base should be great enough to prevent sliding.

PROBLEMS

41. Given the dam having the dimensions and loading indicated in Fig. 29. Consider concrete to weigh 150 lb. per cu. ft. (*a*) Find the horizontal and vertical components of the water pressure. (*b*) Find the position where the resultant pressure cuts the base. (*c*) Find the stresses at the heel and toe of the dam. (*d*) Would this dam be considered safe?

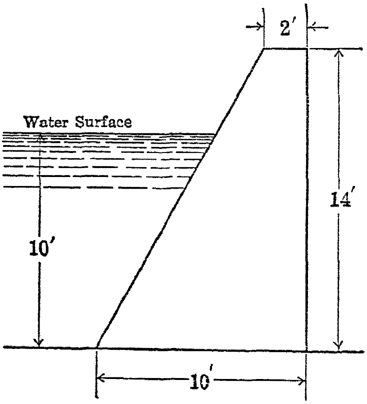


Fig. 29.

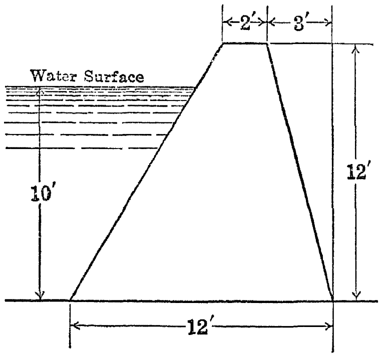


Fig. 30.

42. Given the dam having the dimensions and loading indicated in Fig. 30. (*a*) Find the horizontal and vertical components of the water pressure. (*b*) Find the position where the resultant pressure cuts the base. (*c*) Find the stresses at the heel and toe of the dam.

Ans. $F_h = 3{,}120$ lb., $F_y = 1{,}820$ lb.

43. Given the flashboard shown in Fig. 31. Find the depth of water and the compressive force in the strut per foot of length of crest at the instant that the water is just ready to tip the flashboard.

Ans. $F = 2{,}190$ lb.

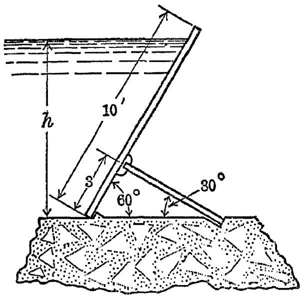


FIG. 31.

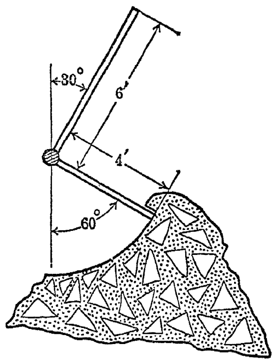


FIG. 32.

44. Given the automatic spillway crest shown in Fig. 32. The weight of the lower leaf is 400 lb. per ft. of length of crest and that of the upper leaf is 500 lb. per ft. of length. Consider these weights concentrated at the center of the leaf. Find the depth of water when the spillway is just ready to tip.

Ans. $h = 4.96$ ft.

27. Pressure Diagrams and the Action of Fluids of Different Specific Gravities. — When areas subjected to fluid pressures are rectangular with one side horizontal the resultant pressure is often most easily found by the

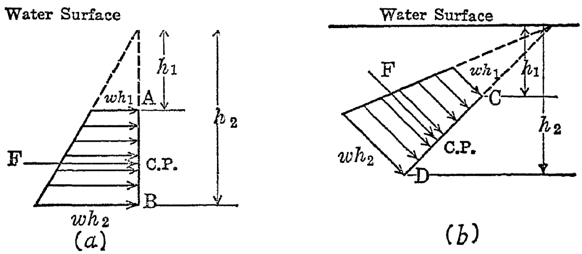


FIG. 33.

use of pressure diagrams. A pressure diagram is a graphical representation of the variation in intensity of pressure over the area. Figure 33*a* shows the pressure diagram for a vertical rectangular area, *AB*. Since *A* is h_1

feet below the surface the intensity of pressure at A may be represented by the ordinate wh_1. The pressure ordinate at B would be wh_2 and the variation between A and B would be linear. The total pressure acting against AB per foot of width is equal to the area of pressure diagram and the center of pressure may be found by locating the centroid of the pressure diagram.

Figure 33*b* shows the pressure diagram for an inclined rectangular area *CD*.

An example of the use of pressure diagrams is given in the following illustrative problem.

Illustrative Problem: Let it be required to find the magnitude of the force exerted upon the side of a box tank which is 2 ft. square and 4 ft. deep when filled one-half full with a liquid having a specific gravity of two while the remainder is filled with a liquid having a specific gravity of one.

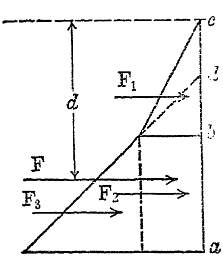


FIG. 34.

The two liquids will be treated separately. The pressure diagram will be as shown in Fig. 34. The intensity of pressure acting at b is $(2)(62.4) = 124.8$ lb. per sq. ft., while that acting at a is $124.8 + (2)(124.8) = 374.4$ lb. per sq. ft.

$$F_1 = \tfrac{1}{2}(124.8)(4) = 249.6 \text{ lb.}$$
$$F_2 = (124.8)(4) = 499.2 \text{ lb.}$$
$$F_3 = \tfrac{1}{2}(249.6)(4) = 499.2 \text{ lb.}$$

The three forces act at the centroids of the respective pressure diagrams. The position of F, which is the resultant of the three forces, may be found by taking moments.

$$Fd = 249.6(1.33) + 499.2(3) + 499.2(3.33)$$
$$d = \frac{3494}{1248} = 2.8 \text{ ft.}$$

The force F is equal to 1248 lb. and acts 2.8 ft. below the surface of the liquid.

This same problem will now be solved by the use of equations (17) and (20) in order to illustrate the meaning of a free surface when the pressure on the surface of a given liquid is not atmospheric. The free surface is considered to be at the elevation at which the pressure would have been atmospheric had the given liquid extended to that elevation. The equivalent free surface for the heavier liquid illustrated in Fig. 34 would have been at d which is 3 ft. above the bottom of the box. The value of the force acting upon the area ab could have been found in one step by considering point d as the elevation of the free surface. The resultant of F_2 and F_3 of Fig. 34 would have been found in this case and its position would fall between the

two forces. We will now proceed to find this force.

$$F = w\bar{h}A = 124.8(2)(4) = 998.4 \text{ lb.}$$

$$y_p(\text{measured from one ft. below } c) = 2 + \frac{0.333}{2}$$

$$= 2.167 \text{ ft.}$$

$$\text{Total force equals } 998.4 + 249.6 = 1248 \text{ lb.}$$

$$d = \frac{249.6(1.33) + 998.4(3.167)}{1248} = 2.8 \text{ ft.}$$

While the results obtained by the two methods agree, it is evident that the amount of labor involved in making the second solution is much greater than that of the first. It is always desirable to consider ease of solution.

PROBLEM

45. A tank is 3 ft. square and 6 ft. deep. The lower 2 ft. is filled with a liquid having a specific gravity of 2; the remainder of the tank is filled with a liquid having a specific gravity of 1.2. The side is held on by means of a bolt at each corner. Find the force in a lower bolt and the force in an upper bolt.

Ans. $T_1 = 1480$ lb., $T_2 = 691$ lb.

28. Stresses in Thin Cylindrical Shells Due to Fluid Pressure. — A thin cylindrical shell is one in which the thickness of the shell is small in comparison with the diameter. Pipes carrying fluids and tanks used for storing fluids may usually be considered as thin shells.

Figure 35*a* represents the cross-section of a thin-walled cylinder subjected to an internal fluid pressure. The intensity of fluid pressure, p, within the cylinder will be assumed constant although there may be a small

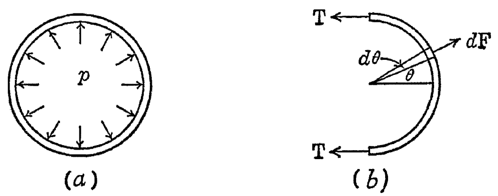


FIG. 35.

variation in this intensity from point to point because of a difference in elevation. This variation is usually small in comparison with the pressure and no serious error will result if p is taken equal to the intensity of pressure at the center of the cylinder. The length of the cylinder considered is l and its diameter d.

If we assume a longitudinal section to be passed cutting the cylinder in half, the forces holding one-half of the cylinder in equilibrium are as shown in Fig. 35*b*.

T is the internal tensile force in the wall of the cylinder acting on a longitudinal section of length l. The radial forces shown act on differential elements of the inside surface and are equal to pdA. Since the width of the element of area, dA, along the arc is $rd\theta$ and its length is l, dA equals $rld\theta$ and the force acting on any element at an angular distance θ from the horizontal axis is

$$dF = prld\theta$$

Summing forces in a horizontal direction, we obtain

$$2T = \int_{-(\pi/2)}^{+(\pi/2)} prld\theta \cos\theta$$

$$= 2prl = pdl$$

or

$$T = \frac{pdl}{2}$$

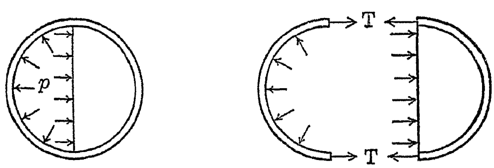
In this equation T is the tension in pounds when p is in pounds per square inch, d is in inches, and l is in inches.

If the thickness of the wall is small, the unit tensile stress in the wall is uniform and may be obtained by dividing T by the area on which it acts. Calling t the thickness of the shell, the area on which T acts is tl sq. in. and the unit stress in pounds per square inch is

$$s_t = \frac{T}{A} = \frac{T}{t \times l} \quad \frac{pd}{2t} \tag{25}$$

Equation (25) can be derived by a simpler method than that given above.

For convenience, imagine a thin membrane stretched across the diameter of the cylinder so as to divide the interior into two equal parts. The



(*a*) (*b*)

FIG. 36.

forces exerted by the fluid on the membrane and on one-half of the cylinder will then be as shown in Fig. 36*a*. If the right and left halves of the cylinder are separated as in Fig. 36*b* the force T may be computed by

applying the equations of equilibrium to either half. Choosing the one to the right, we obtain

$$2T = p \times d \times l$$

or

$$T = \frac{pdl}{2}$$

The student should note that $d \times l$ is the projected area of one-half of the cylinder on a plane at right angles to the direction of the summation of forces and that *the resultant force exerted by the fluid on the curved surface is equal to p times the projected area.*

The unit tensile stress may be found by the use of the equation

$$s_t = \frac{pd}{2t} \tag{25}$$

as before.

Illustrative Problem: Find the minimum thickness of wall for a standpipe which is 6 ft. in diameter and is 40 ft. high, and is to be filled with water. Allowable unit stress in the steel is 16,000 lb. per sq. in.

$$t = \frac{pd}{2s} = \frac{17.32(72)}{2(16{,}000)} = 0.039 \text{ in.}$$

The above problem illustrates that, from the standpoint of the unit stress in the metal, the thickness of the metal need not be very great. The thickness is not alone governed by a consideration of the unit stress which is caused by the internal pressure. Such factors as the action of wind, need of joints, rigidity to permit shipping and handling, and deterioration due to corrosion must be considered.

PROBLEMS

46. A tank is 4 ft. in diameter and 10 ft. high. The tank is built of staves and is held together by means of a hoop at the bottom and one at the top. The tank is one-half filled with water. Find the tension in the hoops.

47. Allowing 16,000 lb. per sq. in. in the shell of the pipe, find the minimum thickness for a pipe carrying 150 lb. per sq. in. if it is 48 in. in diameter.

48. Find the maximum spacing of hoops which go around a 6-ft. diameter stave pipe which carries water under a pressure of 300 lb. per sq. in. The hoops are $\frac{3}{4}$ inch in diameter and the allowable unit stress is 20,000 lb. per sq. in.

Ans. $s = 0.82$ in.

49. If the hoops were touching each other on the pipe described in Prob. 48, what would be the maximum allowable head that could be placed on the pipe?

CHAPTER IV

VISCOSITY

29. The Meaning of Viscosity.—All bodies, whether solids, liquids or gases, offer a resistance to deformation or relative displacement of the portions of the body against one another. This resistance may be of different kinds; but for liquids and gases, it may increase as the velocity with which parallel planes a fixed distance apart in the fluid increases relative to each other. In this case, the force is due to the property of the fluid known as *viscosity*.

From the above, it can be seen that viscosity is a property of a fluid which can be discerned only when motion takes place between the different parts of the fluid body. It is common knowledge that the resistance offered by different fluids is not the same. These facts can be demonstrated by placing a paddle in a pool of water and then pulling it in a direction parallel to the plane of the blade, first at a slow velocity and then at a high one. The difference in force required would be appreciable. Now place this same paddle in a pool of thick oil and produce the same type of motions. The forces required in the latter case would be considerably different from those in the former since the viscosity of the oil is much higher.

30. Historical Sketch.—Newton was the first to study the action of viscosity with sufficient seriousness to arrive at a hypothesis as to the magnitude of the force required to overcome viscous resistance. This work was published in his "Principia," 2nd Ed., 1713. He described viscosity as a "lack of slipperiness" between the particles of the fluid. He used as a hypothesis "That the resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated one from another." This assumption is equivalent to saying that if two layers of fluid are separated from each other the force required to maintain a relative velocity with reference to the two layers would be

$$F = \mu A \frac{dV}{dy} \tag{26}$$

where μ is a constant, or the viscosity, for each fluid. From Newton's hypothesis and from Eq. (26) it is evident that with two plates, such as AB and CD of Fig. 37, in which AB is stationary and CD is moving with

a velocity V, there would be a straight line variation in the velocity of the fluid between the two plates. However, for this relationship to hold, the distance between the plates would need to be held small and the velocities V could not exceed a certain critical value.

In spite of this early recognition of the existence of an internal friction within the fluid, the hydrodynamics as developed by Daniel Bernoulli, Euler, and others considered only frictionless liquids. Simple experiments demonstrated that the behavior of the fluids departed considerably from that which would be predicted from a consideration of the formulas. Bernoulli attributed these large differences to the adhesion of the liquid to the sides of the tube. The many experiments conducted by the French engineers about 1800 brought out many facts that hydrodynamics failed to explain. It was because of this failure of hydrodynamics to produce usable formulas that engineers came to depend more and more upon the empirical formulas which were based upon their experiments.

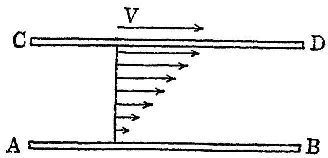


FIG. 37.

In the early experiments, it was found that the resistance to flow was a function of the sum of two terms, one a first power of the velocity and the other a second power. In 1839, Hagen studied the flow through brass tubes whose radius varied from 0.127 cm. to 0.294 cm. He found that the quantity discharged through these tubes in a unit time was directly proportional to the pressure and the fourth power of the radius, and inversely proportional to the length of the tube. He also found departures from this law which he ascribed to turbulence and that this turbulence set in more easily at high temperatures.

The work of Hagen was followed shortly by that of Poiseuille, a French physician, who was interested in the flow of blood through the capillary system of the human body. He used very fine glass capillary tubes which had diameters ranging from 0.03 to 0.14 mm. Many measurements were made in which only one factor was varied at a time. From these experiments, Poiseuille concluded, (1) that the quantity discharged in a unit of time was proportional to the pressure, provided that the length of the tube exceeded a certain minimum which increased with the size of the tube, (2) the discharge was inversely proportional to the length of the tube, and (3) the discharge was directly proportional to the fourth power of the diameter.

The discharge could therefore be expressed by the equation

$$Q = k\frac{PD^4}{L} \tag{27}$$

where k was a characteristic of the liquid, the value of which increased with the temperature. He studied the manner of the variation.

Later investigators studied the meaning of the k in Poiseuille's equation and obtained the equation

$$Q = \frac{\pi P R^4}{8 \mu L} \tag{28}$$

in which μ was the coefficient which we know as coefficient of viscosity. In the c.g.s. system, the unit of viscosity can be taken from the definition formula, Eq. (26), and is 1 dyne per sq. cm. for 1 cm. per sec. per cm.; or it is 1 dyne sec. per sq. cm. This unit has been called a "poise" in honor of Poiseuille. The viscosity of water at room temperature is just about 0.01 poise, or one centipoise. For this reason, the viscosity is often given in centipoises. One hundred centipoises equal a poise. Equation (28) is known as Poiseuille's equation of flow, despite the fact that it was not given in that form by him. The derivation of Poiseuille's equation will be found in the chapter on flow through pipes.

During the following years, the viscosity of many liquids was measured, but the real next step was made by Osborne Reynolds (see Art. 78 for a discussion of Reynolds' work) who definitely established, in his paper which was published in 1883, the fact which Hagen had suspected in 1839, namely: that two types of flow existed. He found, however, that the criterion for the change from one type of flow to the other was not a function of the viscosity alone, but of the viscosity divided by the density of the fluid. This ratio of viscosity to the density is called "*kinematic viscosity*."

31. Variation of Viscosity with Temperature. — For a given pressure, the viscosity of liquids decreases with an increase in temperature. The decrease per degree is much greater at low than at high temperatures. This condition is illustrated by Fig. 38. No general law has been found by which the viscosity can be expressed in terms of temperature, although for any one liquid, the variation can be represented with a fair degree of accuracy by empirical formulas.

The viscosity of gases can best be explained by a consideration of the molecular activity for the given condition. The molecules of the gas move about with a high velocity and as a given molecule moves from a region of low velocity to one of high, or the opposite, there is an interchange of momentum. In all cases involving impact of particles such that the law of conservation of momentum applies, there is always a loss of energy in the system. The viscous drag offered by gases is considered to be of this type.

As the temperature of the gas rises, the molecular activity becomes

greater and the number of impacts increases. As a consequence, the viscosity of the gases increases with rising temperatures.

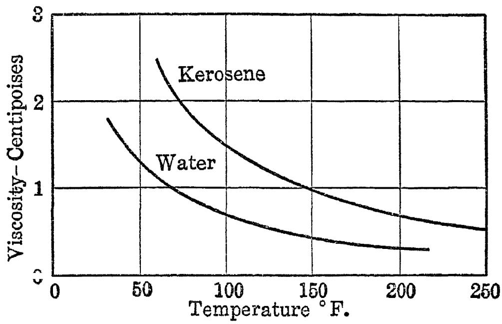


FIG. 38. Decrease in viscosity caused by an increase in temperature.

32. Variation of Viscosity with Pressure. — The viscosity of all liquids examined, except water, increased with pressure and may become very high for high pressures. This was demonstrated by Bridgman,[1] who investigated a number of liquids with pressures ranging up to 12,000 atmospheres at temperatures of 30° and 75° C. For some liquids, the effect of temperature was small; but for others, the viscosity at the high temperature was about 1000 times as great as at ordinary pressures.

Water behaves in a peculiar manner in this respect as it does in respect to many of its other properties. At temperatures below 30° C., the viscosity decreases for increasing pressure until a pressure of about 1000 atmospheres has been reached after which it increases. The lower the temperature, the more marked is the minimum viscosity. For temperatures above 30° C., water behaves like other liquids in that the viscosity increases throughout the entire range of increasing pressure.

Liquids having much the same characteristics at ordinary pressures do not necessarily respond to the application of pressure in the same way. This fact has been well illustrated by tests[2] made on a Pennsylvania and a California oil which had nearly identical viscosities at atmospheric pressure condition whether at a temperature of 130° F. or of 210.2° F.

Maxwell, in developing the kinetic theory of gases, derived an expression from which it followed that the viscosity of the gas was independent of the pressure. This conclusion was tested experimentally and it was

[1] "The Effect of Pressure on the Viscosity of Forty-three Pure Liquids." Bridgman. *Proc. Am. Acad. of Arts and Sciences*, V. 61., Feb. 1926, pp. 57–99.

[2] "Effect of Pressure on the Viscosity of Oils and Chlorinated Diphenyls." Dow, Fenske and Morgan. *Ind. and Eng. Chem.*, V. 29, Sept. 1937, pp. 1078–80.

found that the viscosity was constant for pressures between 760 and 1 mm. of mercury. There is a considerable deviation from this law at high pressures; and at very low pressures where the free path of the molecules becomes great, the viscosity diminishes considerably.

The effect of the pressure upon the viscosity of the Pennsylvania and the California oils is shown graphically in Fig. 39.

33. Types of Viscosimeters in Use. — Instruments for measuring viscosity are called viscosimeters. For purposes of discussion, these instruments will be divided into two classes, namely: the scientific and

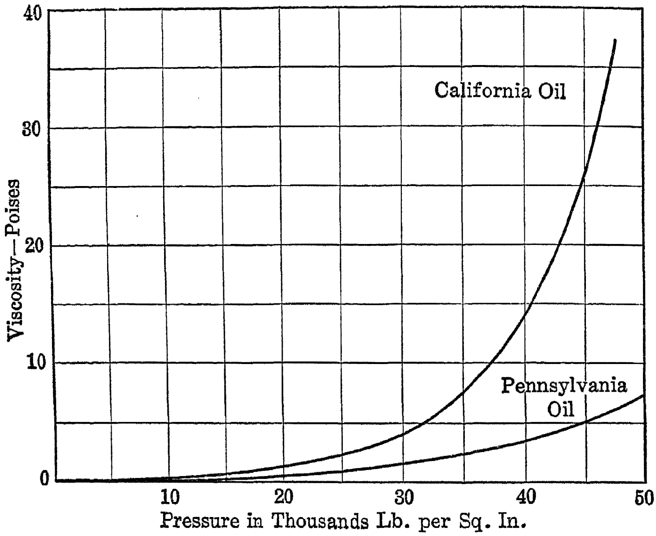


Fig. 39. The effect of pressure upon the viscosity of oil.

the technical types. The scientific type includes all instruments in which the viscosity is measured by the flow through a capillary tube, while the technical type will be any other in which the viscosity is not so measured.

There are a number of the capillary types of viscosimeters, but two will be described.

The Thorpe and Rodger instrument (see Fig. 40) consists of the capillary tube *CD* whose diameter and length are accurately known. A definite volume of liquid is placed in the right-hand leg and air pressure is then applied to the left-hand leg until the liquid stands at k_1, any excess overflowing into the trap T_2. A known pressure which is measured by a water manometer is then applied to the right leg, and the liquid forced

down, and the time required for the liquid to drop from m_3 to m_4 taken on a stop watch. The liquid is then forced up the left leg until the level is at k_2, after which the same pressure is applied to the left leg and the time for the liquid to drop from m_1 to m_2 again measured. Volume L equals volume R, and the two times are averaged. The viscosity is then computed from Poiseuille's equation.

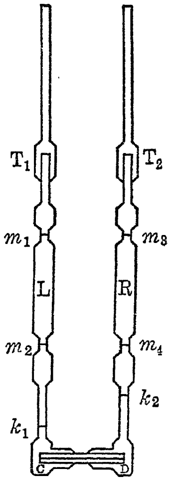


FIG. 40. The Thorpe and Rodgers Viscosimeter

Another capillary tube viscosimeter is that designed by Ostwald in which the pressure causing flow is produced by the weight of the liquid itself (see Fig. 41). In this instrument, a constant quantity of liquid is placed in the wide right leg and then drawn through the capillary into the bulb and well above the mark A. It is then allowed to drain back, and the time between the marks A and B is taken. This is done for a standard liquid, probably a sugar solution of known concentration, of known viscosity μ_0 and density ρ_0. These readings are taken several times at a convenient temperature and the average time t_0 used.

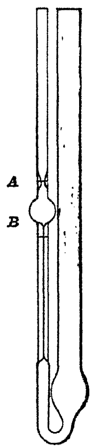


FIG. 41. The Ostwald Viscosimeter. (Courtesy Eimer and Amend.)

Any other liquid of density ρ_1 whose viscosity is desired is then placed in the instrument and the time t_1 is obtained. The viscosity μ_1 of this liquid is then obtained from the equation

$$\mu_1 = \mu_0 \frac{\rho_1 t_1}{\rho_0 t_0} \tag{29}$$

A number of tubes which are fundamentally similar to the Ostwald viscosimeter have been produced. These other tubes have been designed in an attempt to eliminate certain defects that are inherent in the Ostwald instrument. All instruments of this type obtain the desired viscosity by comparison with the previously measured viscosity of some standard liquid. The viscosity is not obtained directly.

The technical type of viscosimeter is the one which would normally be found in the ordinary laboratory. It is necessary for an instrument of this type to be calibrated, and then for other instruments of the same class to be made similarly so that they give the same readings as the calibrated one. Three fundamentally different instruments of this group will be described. These are the short tube, the torque, and the resistance viscosimeters.

Of the short tube type, the more common ones are the Saybolt Universal, or Saybolt Furol, the Engler, and the Redwood. The Saybolt Universal and the Saybolt Furol are similar with the exception that the short tube through which flow takes place in the Furol has a larger diameter than that in the Universal. A thicker, more viscous fluid will flow through this larger tube in a reasonable time. The Saybolt instruments are most commonly used in the United States, the Engler in Germany, and the Redwood in Great Britain.

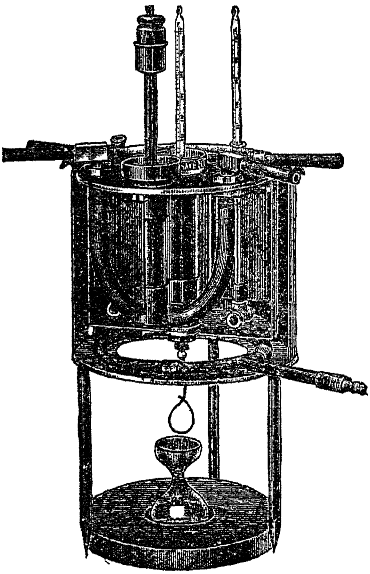


Fig. 42. The Saybolt Universal Viscosimeter. (Courtesy Eimer and Amend.)

The Saybolt Universal viscosimeter is shown in Fig. 42. It was developed mainly for the measurement of the viscosity of oils and consists of the oil tube, bath, receiver, thermometers, timer, and withdrawal tube. The oil tube, shown in Fig. 43, is the essential element and is made entirely of a corrosion resistant metal. The inside diameter of the outlet tube is 0.1765 ± 0.0015 cm. and the length of the tube is 1.225 ± 0.010 cm. Surrounding the oil tube is the bath which has the dual purpose of offering a support for the oil tube, and serves as a container for the bath liquid. It must contain a stirring device for obtaining uniform temperatures, and means for heating or cooling. The receiving flask is of glass and has a capacity of 60 ml. The time for obtaining this discharge is taken with a stop-watch.

The Saybolt Universal viscosimeter can only be used if the time required for the discharge of the 60 ml. of fluid is 32 sec. or more. Should the time of efflux become excessive (more than 1000 or 2000 sec.), the Saybolt Furol viscosimeter may be used. The Furol is similar to the Universal except that the inside diameter of the outlet tube is $0.315 \pm .002$ cm. and its time of efflux is about one-tenth that of the Universal. The Furol is not recommended for use where the time of efflux is less than 25 sec. The test procedure for the Saybolt instruments is specified in the *A.S.T.M. Standards*, Sec. D–2.

The value of the kinematic viscosity is obtained by the use of the dif-

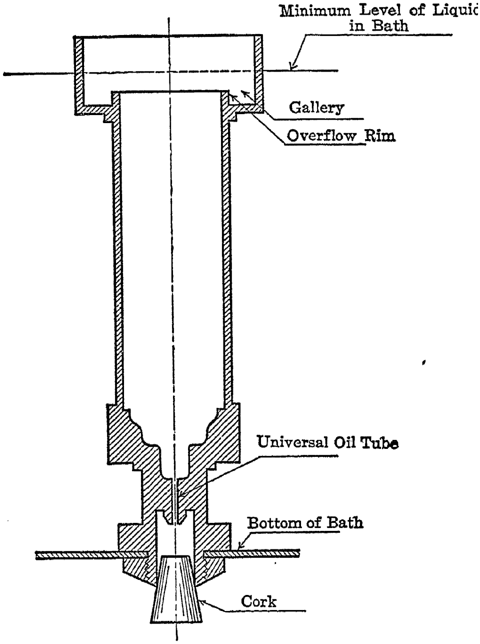


FIG. 43. The Saybolt Oil Tube.

ferent short tube types of viscosimeters. The equations for these instruments follow:

Engler $$\frac{\mu}{\rho} = 0.00147t - \frac{3.74}{t} \tag{30}$$

Redwood $$\frac{\mu}{\rho} = 0.0026t - \frac{1.715}{t} \tag{31}$$

Saybolt Universal $$\frac{\mu}{\rho} = 0.0022t - \frac{1.80}{t} \tag{32}$$

Saybolt Furol $$\frac{\mu}{\rho} = 0.0222t - \frac{2.03}{t} \tag{33}$$

where μ = viscosity in poises,

ρ = density (grams per cubic centimeter),

t = time of efflux in seconds.

Two common viscosimeters of the torque type are the MacMichael and Stormer instruments. The Stormer is illustrated in Fig. 44. The viscosity is determined by measurement of the time required for a definite number of revolutions of a rotating cylinder, or other type rotor, which is immersed in the liquid being tested. The sample is maintained at the desired temperature by means of a water or oil bath. The rotor is driven by means of a definite weight. A revolution counter is attached to the spindle of the rotor.

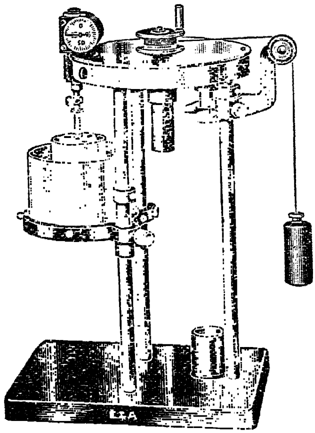


FIG. 44. Stormer Viscosimeter. (Courtesy of Eimer and Amend.)

The relative viscosity is obtained by dividing the time required for the rotor to make a specified number of revolutions in the test sample by the time required for the cylinder to make the same number of revolutions in distilled water, or other standard reference, using identical procedure, temperature, and weight.

The viscosities are obtained in the absolute units, i.e., the poise, by means of a calibration table. The readings obtained by the torque type instruments are not dependent on the specific gravity of the test sample. This was not true with the short tube types.

The viscosimeters, which are here classified as the resistance type, are very simple. One might be of the bubble type, Fig. 45, which consists of a number of viscosity tubes containing a mineral oil that will not change in viscosity with time. The approximate viscosity for each oil in poises at 25° C. is given. For conducting a test, the bubble of the sample is first adjusted to be approximately the size of that in the standard tube. The tubes are then brought to a temperature of 25° C. and the viscosity is determined by locating the standard tube in which the speed of the bubble is the same as that in the tube which contains the sample.

Another method appearing under this same category would be to place the sample in a tube and note the time required for spheres of known size and density to fall a given distance.

34. Conversion of Viscosity from the C.G.S. to the English System of Units. — The viscosities of various fluids are normally published in terms of the poise which is in the c.g.s. system. Referring to the equation

$$F = \mu A \frac{\Delta V}{\Delta y}$$

we have the different variables expressed as

$$\text{dynes} = \text{poise} \times \text{cm.}^2 \times \frac{\text{cm. per sec.}}{\text{cm.}}$$

It follows that the unit of the poise is

$$\text{dynes sec. per cm.}^2$$

When the ft.-lb.-sec. system is used, the units of the absolute viscosity, μ, must be converted to lb. sec. per ft.^{2}

$$1 \text{ gram} = 980.7 \text{ dynes}$$

$$1 \text{ lb.} = 453.6 \text{ grams}$$

$$1 \text{ ft.} = 30.48 \text{ cm.}$$

The absolute viscosity in poises can therefore be converted to the absolute viscosity in the English system by multiplying the poise by the factor

$$\frac{(30.48)^2}{(980.7)(453.6)} = 0.00209$$

The absolute viscosity in the English system has no standard accepted name.

As stated in Art. 30, the behavior of the fluid may be dependent upon the kinematic viscosity rather than upon the absolute viscosity. The

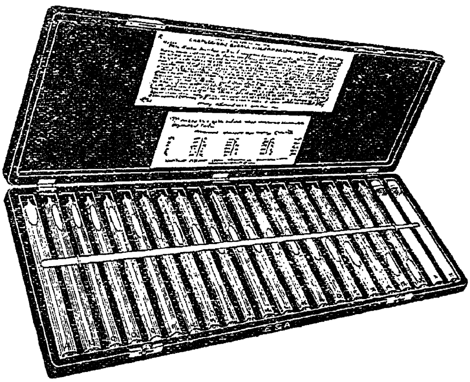
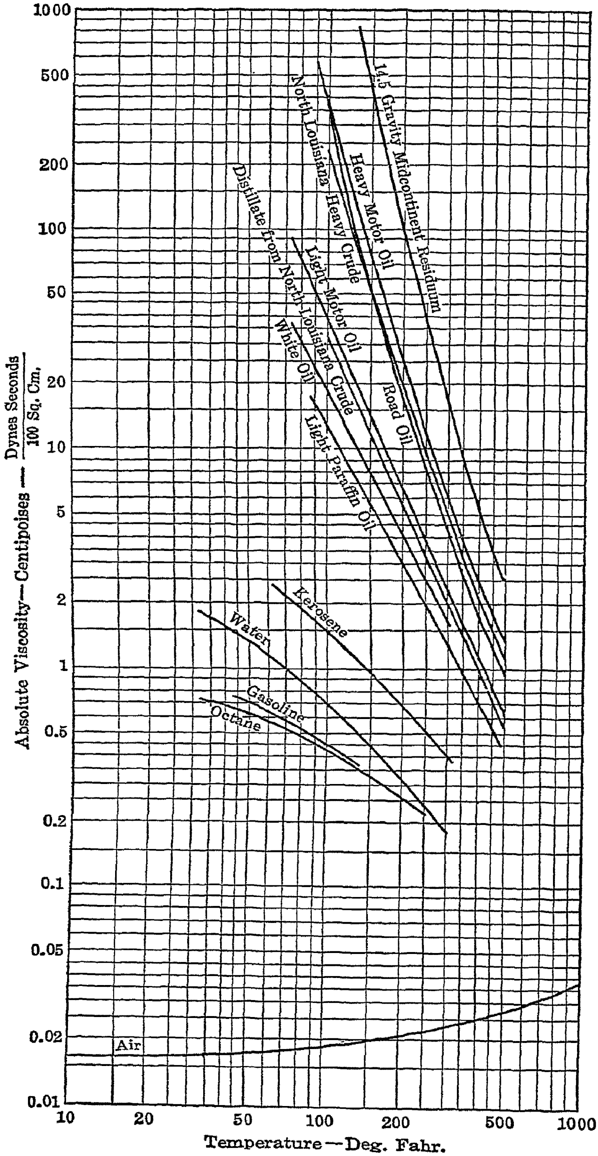


FIG. 45. Gardner-Holdt Bubble Viscosimeter.
(Courtesy Eimer and Amend.)

kinematic viscosity of a fluid is found by dividing the absolute viscosity by the density of the fluid. In the c.g.s. system, sufficient accuracy will normally be obtained by taking the density as equal to specific



1000
500
200
100
50
20
10
5
2
1
0.5
0.2
0.1
0.05
0.02
0.01
10
20
50
100
200
500
1000
Absolute Viscosity—Centipoises — Dynes Seconds / 100 Sq. Cm.
Temperature—Deg. Fahr.
14.5 Gravity Midcontinent Residuum
Heavy Motor Oil
North Louisiana Heavy Crude
Distillate from North Louisiana Crude
Light Motor Oil
White Oil
Road Oil
Light Paraffin Oil
Kerosene
Water
Gasoline
Octane
Air

FIG. 46

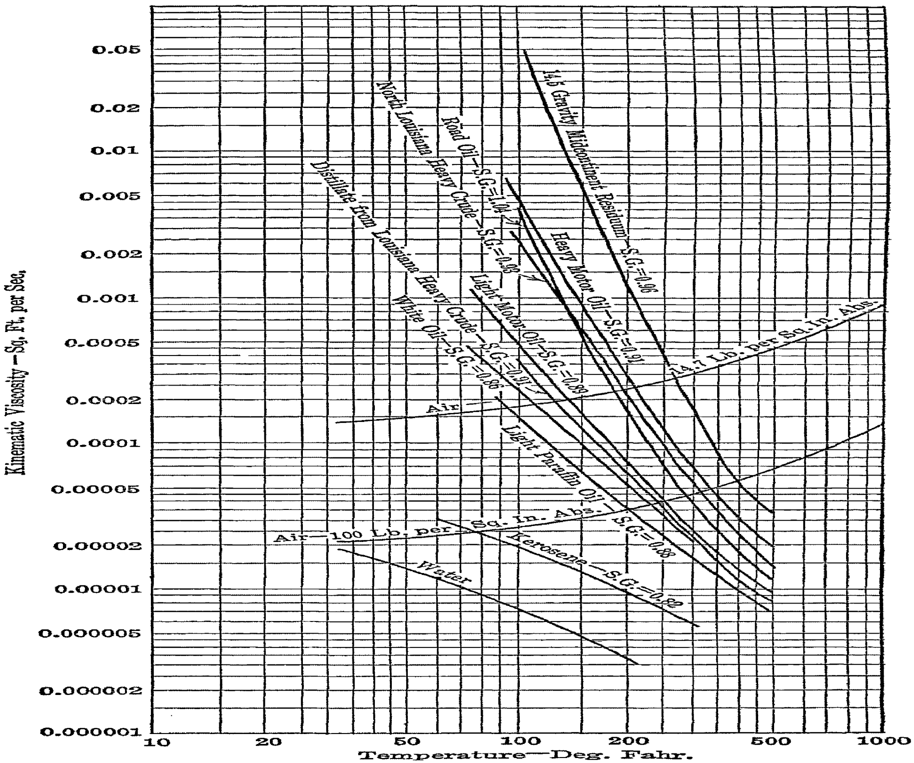

Kinematic Viscosity—Sq. Ft. per Sec.
0.05
0.02
0.01
0.005
0.002
0.001
0.0005
0.0002
0.0001
0.00005
0.00002
0.00001
0.000005
0.000002
0.000001
10
20
50
100
200
500
1000
Temperature—Deg. Fahr.
14.5 Gravity Midcontinent Residuum—S.G.=0.96
North Louisiana Heavy Crude—S.G.=0.93
Road Oil—S.G.=1.04
Distillate from Louisiana Heavy Crude—S.G.=0.91
Heavy Motor Oil—S.G.=0.91
Light Motor Oil—S.G.=0.89
White Oil—S.G.=0.86
14.7 Lb. per Sq. In. Abs.
Air
Light Paraffin Oil—S.G.=0.88
Air—100 Lb. per Sq. In. Abs.
Kerosene—S.G.=0.82
Water

FIG. 47

gravity. In the English system, the density is equal to the weight of the fluid in pounds per cubic foot divided by the acceleration due to the pull of gravity which will be taken equal to 32.16 ft. per sec^2. The units of kinematic viscosity in the c.g.s. system are square centimeters per second, while those in the English system are square feet per second.

The unit of kinematic viscosity in the c.g.s. system is sometimes called the stoke, while there is no name for the corresponding unit in the English system.

The absolute and kinematic viscosities of a number of fluids[1] are given in Figs. 46 and 47 respectively. Reference is to be made to these charts for the viscosities of the fluids mentioned in the problems throughout the remainder of this text.

PROBLEMS

50. Two plates having dimensions of 2 ft. $\times$ 4 ft. are separated by a distance of $\frac{1}{2}$ in. One plate is moving at 3 ft. per sec. with respect to the other. Find the force in pounds required if the space between the plates contains the heavy motor oil of Fig. 46 at a temperature of 100° F.

51. Water at 75° F. was used as a standard liquid in an Ostwald viscosimeter and the time of discharge was 80 sec. An oil which weighs 0.85 gr. per cu. cm. was used in the same tube and the time was 500 sec. Find the viscosity of the oil in centipoises.

52. An oil is tested in a Saybolt Universal viscosimeter and the time of efflux was 1100 sec. The oil weighs 0.89 gr. per cu. cm. (*a*) Find the viscosity in poises. (*b*) Find the absolute viscosity in the English units.

Ans. (*a*) $\mu = 2.15$ poises, (*b*) $\mu = 0.00448$ lb. sec. per sq. ft.

53. Find the time of efflux for an oil similar to that of Prob. 52 if tested in (*a*) the Engler, (*b*) the Redwood, and (*c*) the Furol viscosimeters.

[1] For an excellent discussion on the viscosity of oils at high temperatures, see "Viscosity of Oils at High Temperatures." Fortsch and Wilson. *Ind. and Eng. Chem.*, V. 16, Aug. 1924, pp. 789–92.

CHAPTER V

TYPES OF MOTION. BERNOULLI'S THEOREM. FORMS OF ENERGY.

35. Introduction. — The study of fluids at rest involves, fundamentally, the determination of the pressure variation throughout the fluid. Ordinarily, the pressure exerted upon any particle within a fluid at rest can be found with certainty. When one knows the pressure variation, it is a simple matter to apply the equations of mechanics in order to obtain the information desired, such as the resultant force upon a submerged surface or body. No simplifying assumptions are necessary, and the final results are rarely in question.

The study of fluids in motion is much more complicated than that of fluids at rest for two basic reasons. In the first place, every particle in the moving fluid is acted upon not only by forces perpendicular to any plane passing through the particle, but also by shearing forces acting tangentially. These shearing forces come into play whenever relative motion between particles of the fluid occurs, and are due to that property of any real fluid called viscosity. An elementary volume of fluid is shown in Fig. 48 with the stresses which act on three of the faces indicated. The other three faces are subjected to similar stresses. Each face of the cube has a normal stress or pressure, P, acting on it, and two shearing stresses, τ. In general, these stresses are not the same on the different faces and vary as the particle moves from one point to another. Furthermore, successive particles occupying the space shown might have stresses differing from those indicated in the figure.

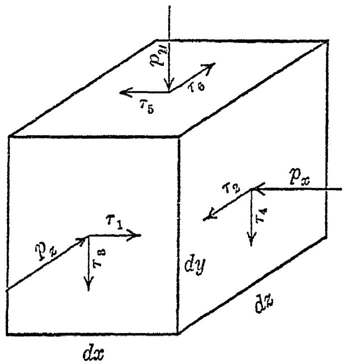


FIG. 48

Secondly, the difficulties of the problem are greatly increased because of the motion of the individual particles themselves. Complete knowledge would require that the magnitude and direction of the velocity of every particle at every instant be known.

The student should need no further proof than this to convince him that the complete solution of the general case of a fluid in motion is an exceedingly complicated one, and should realize the necessity of using all the simplifying assumptions consistent with practical results. The proof that certain assumptions are permissible can come only from a comparison of the results obtained from theory with those obtained by experiment. In this text every attempt will be made to state clearly the assumptions used in any derivation, and the limitations imposed by such assumptions.

36. Types of Flow. — Observations of fluids in motion have disclosed several well defined types of flow. While some of these types have been analyzed mathematically with considerable success, the others have presented insurmountable mathematical obstacles. Consequently, in addi-

(a) (b)

FIG. 49. Streamline flow.

tion to the observed types, it has been necessary to invent fictitious flows which would be possible only with the non-existent perfect or frictionless fluid. However, experiments have demonstrated that conclusions drawn from a study of such imaginary types of flow are, in many cases, directly applicable to the flow of real fluids, or may be easily modified to take care of the neglected properties. In other cases, the assumption that a fluid is frictionless may lead to results which are very much in error.

Classification of the various types of flow is usually made by describing the motion of the particles of the fluid. This description can be made in one of two ways. Attention can be directed to the path of the individual particles, or conditions at any point in the fluid may be studied as the space at the point is occupied by successive particles. Definitions of some of the more common types of flow in accordance with these two methods of description will follow. Discussion of the conditions under which the various types of flow may be expected is left to later chapters where they are studied in greater detail.

Streamline, or *Viscous*, flow is one in which there is no intertwining of the paths of the particles. In this type of flow the paths of the particles of a fluid moving past a submerged plate would appear as shown in Fig. 49*a*, and as in Fig. 49*b* for the same type of flow in a pipe. These same

streamlines would be obtained by hydrodynamics for a non-viscous fluid in which there would be no friction. Such fluids do not exist.

Laminar flow is a particular type of viscous flow occurring when the particles of a fluid move in straight lines as in a pipe of uniform diameter, or for flow between parallel plates.

Turbulent, or *Sinuous*, flow is characterized by an almost complete absence of order in the paths taken by the individual particles. The particles move helter skelter, here and yon, within the fluid, and it is impossible to trace the path of an individual particle. Some flows are more turbulent than others, and a satisfactory scale for degree of turbulence is difficult to attain. Unfortunately this type of flow is by far the most common and the most difficult to analyze.

Steady flow is a flow in which the same weight of fluid flows past any given section in a given unit of time, in other words, the discharge is constant. By discharge is meant the weight of flow per unit of time.

No flow is steady when it first starts. A certain period of time is required before the flow becomes steady and uniform. Even after flow has become well established, there are still pulsations and variations. This is evidenced in a pipe by variations in the pressure at a given section, as can be seen on most pressure measuring devices; or by the surges in the water surface on a river. While these fluctuations are known to exist, they are not considered inconsistent with the engineering idea of steady flow.

Uniform flow occurs when the cross-sectional area of the stream and the velocity distribution in the stream are the same at all points. Thus, there could be steady uniform flow in a pipeline having a constant diameter, but there would not be steady flow on a steep slope where the fluid was accelerating. This second condition could be steady, but it would be *non-uniform* flow. With gases, it is doubtful whether uniform flow can occur because of the expansion occurring with the changes in pressure which accompany the flow.

37. Streamlines and Stream Tubes. — In some types of flow it is possible to obtain a graphical representation of the flow by drawing what are called streamlines and stream tubes. This method of delineation is especially useful in flows which are not confined within solid boundaries, as in the flow of air past the wing of an aeroplane.

A streamline is a line drawn within a fluid across which no flow occurs. Streamlines are most often drawn so that the spacing between the lines is inversely proportional to the velocity between the lines. Since in unsteady flow the velocity at a point in the fluid generally changes in both magnitude and direction, a streamline pattern drawn for conditions at one instant would differ from that for some other instant. Consequently, streamlines

are usually drawn for steady flow. In this type of flow the streamlines may be taken as the actual path of the particles on the line.

Streamlines have been determined analytically for bodies of simple shape submerged in a perfect fluid. The mathematical background required for this analytical treatment is so extensive even for some of the simplest types of flow that such a treatment is considered beyond the scope of this text. The interested student should consult one of the standard books on "Hydrodynamics."[1]

A stream tube is an imaginary tube included within a group of streamlines. Ordinarily the tube is imagined as having a very small cross-sectional

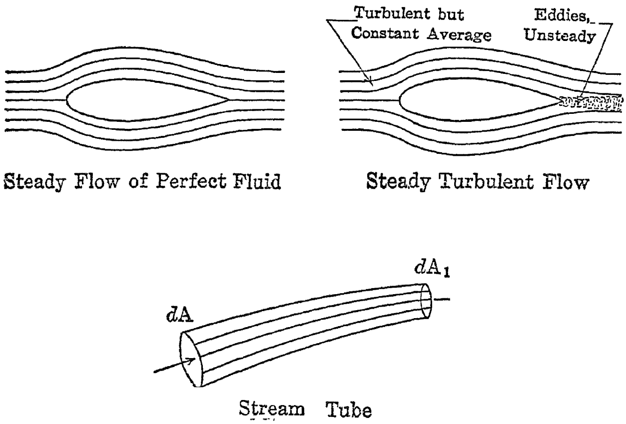


Fig. 50

area so that the velocity within the tube may be considered constant for every point within a cross section. Since a stream tube is bounded by streamlines, no flow can take place through the sides of the tube.

Although the foregoing discussion would apply strictly to a viscous or streamline flow as defined in Art. 36, even in turbulent flow, where a constant average velocity exists, streamlines and stream tubes are helpful concepts. In this case, a fictitious steady flow would replace the actual turbulent flow. However, if the turbulence is such that vortices, eddies, and swirling occur in some part of the fluid, steady flow even in the sense of a constant average velocity at every point cannot exist in this area, and streamline representation is impossible. Such a condition generally occurs at the trailing edge of an airfoil. Figure 50 shows streamlines and stream tubes as discussed in this article.

[1] For a method of determining streamlines experimentally, see "Aerodynamics" by Piercy. Art. 36. D. Van Nostrand Co., 1937.

38. Discharge and Continuity. — The quantity of fluid passing a given cross section of a stream per unit of time is called the discharge at the section. Figure 51 shows a portion of a stream tube. If we consider the cross section dA_1, where the velocity is V_1 and the specific weight is w_1, the volume of fluid passing the cross section per unit of time is V_1dA_1 and the weight of fluid passing per unit time is

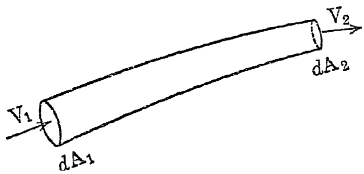


FIG. 51

$$dW_1 = w_1V_1dA_1$$

A stream of finite area would be made up of a large number of stream tubes similar to that in Fig. 50. The discharge through such a stream would then be

$$W_1 = w_1\int V_1dA_1 = w_1A_1\bar{V}_1 \tag{34}$$

where $\bar{V}_1$ is the average velocity in the cross section. Similarly, at another cross section, A_2, the weight discharge is

$$W_2 = w_2A_2\bar{V}_2 \tag{35}$$

In equations (34) and (35) the discharge will be in pounds per second if w is in pounds per cubic foot, A is in square feet, and $\bar{V}$ is the average velocity in feet per second.

When the flow is steady and there is no change in the mass of fluid constantly included between two cross sections, the weight of fluid entering at A_1 is equal to that leaving at A_2. Therefore we can write

$$W = w_1A_1V_1 = w_2A_2V_2 \tag{36}$$

We have dropped the bar over $\bar{V}_1$ and $\bar{V}_2$, indicating the average velocity, and shall consider V_1 and V_2 as average velocities in this equation. Equation (36) is especially useful in dealing with the steady flow of compressible fluids and is called the equation of continuity for compressible fluids.

If the fluid is incompressible, the specific weight is approximately the same at all sections, or

$$w_1 = w_2$$

and

$$Q = A_1V_1 = A_2V_2 \tag{37}$$

In Eq. (37), Q is the volume rate of flow, or discharge, in cubic feet per second, and A and V are the area and average velocity in square feet and feet per second respectively. Equation (37) is the equation of continuity for liquids.

PROBLEMS

54. Water is being discharged through a pipeline at the rate of 2 cu. ft. per sec. Find the average velocity at sections where the diameter is 2 in., 4 in., and 12 in.

55. Air for which R is 53.3 is flowing through a pipeline at the rate of 3 lb. per sec. If the flow takes place at a constant temperature of 80° F., find:

(*a*) The average velocity at a cross section where the gage pressure is 15 lb. per sq. in. and the diameter is 6 in.

(*b*) The average velocity at a cross section where the gage pressure is 10 lb. per sq. in. and the diameter is 10 in.

Ans. (*a*) $V = 103$ ft. per sec., (*b*) $V = 44.6$ ft. per sec.

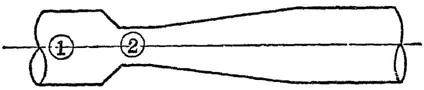


FIG. 52

56. Hydrogen is flowing through the tube shown in Fig. 52 at the rate of 1.2 lb. per sec. At (1) the pressure is 20 lb. per sq. in. gage, the diameter is 8 in., and the temperature 40° F. At (2) the pressure is 15 lb. per sq. in. gage and the diameter is 4 in. Given that hydrogen at 14.7 lb. per sq. in. abs. and at a temperature of 32° F. weighs 0.0057 lb. cu. ft., and assuming the change in the condition between (1) and (2) to be adiabatic, find:

(*a*) The average velocity at (2).

(*b*) The temperature of the gas at (2).

57. The distance between two streamlines drawn for the flow of air past an airfoil varies as the air approaches and passes the airfoil. At a point where the velocity is known to be 100 ft. per sec. the streamlines are 0.5 in. apart. (*a*) What is the average velocity where the streamlines are 0.3 in. apart? (*b*) What distance between streamlines would represent a velocity of 150 ft. per sec.?

39. Pressure in a Moving Fluid. — The study of the motion of fluids requires a clear understanding of what is meant by "static pressure" at a point in the moving fluid, and a discussion of its measurement. Figure 53*a* shows a pipe through which a liquid is flowing. We wish to investigate the pressures existing between particles at a point such as A, at which the velocity is in the direction indicated.

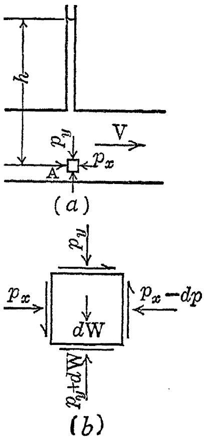


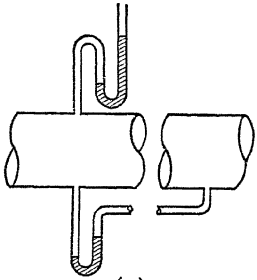
FIG. 53

Figure 53*b* shows a small element of fluid enclosing the point in question with a shearing force and normal force on each of its faces. These forces are exerted upon the given particle by those adjacent to it. If the shearing forces are zero or are small compared with the normal pressures, the values of the two horizontal pressures will be equal. This approximation introduces little error for the drop in pressure, dp, is very small in the length dl. The weight of the fluid, dW, in the elementary volume is a differential of the higher order, so can be neglected.

Thus, p_x and p_y are equal to each other. (Art. 14, p. 16.) This pres-

sure, measured independently of any impact effects arising from the velocity of the fluid itself, is called the static pressure.

The static pressure at a point in a moving fluid, or the difference in pressures existing between two points in the fluid, may be measured by instruments of the kind described for this purpose in the discussion of fluids at rest. Thus an open tube attached to a pipe as shown in Fig. 53*a* could be used to measure the static pressure at a point such as *A*. The distance above *A* to which the liquid would rise would be a measure of the pressure at the point in feet of liquid, and could be converted into any other units of pressure desired. For an accurate measurement of this pressure, the axis of the tube at the pipe wall must be perpendicular to the direction of flow. Otherwise impact forces would arise which would cause erroneous pressure readings. Any burrs or projections near the opening in the wall of the pipe would disturb the flow so that the velocity in this vicinity would not be parallel to the face of the wall. In this case, also, impact effects would give erroneous pressure readings.



(c)

Fig. 53

The simple U-tube, when precautions are observed when connecting it to the pipe wall, can be used to measure relatively high pressures. For differences in pressures, the differential manometer may be used (Fig. 53*c*).

An instrument for measuring the static pressure in a flow which is not confined by solid boundaries is shown in Fig. 54. It consists of a bent tube, one leg of which is closed and shaped so as to create as little disturbance in the flow as possible. This leg contains several holes drilled at right angles to its axis which act as static pressure openings. The same precautions should be taken in drilling these holes that were mentioned in the discussion of the pressure connections in the side of a pipe. The holes should be located at distance equal to several diameters of the tube away from the nose. The long leg of the tube should be at even greater distance than this from the holes so that its influence in disturbing the flow near the holes may be minimized. When a tube of the kind described above is placed so that the axis of the short leg is in the direction of flow, the static pressure in the vicinity of the holes may be measured by means of a manometer attached to the long leg of the tube. A differential manometer connected to two such

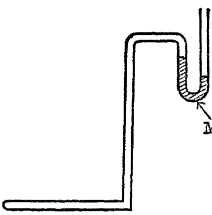


Fig. 54

tubes can be used to measure the difference in the static pressures existing at the two points where the pressure tubes are located.

40. Bernoulli's Equation for Liquids. — In case of steady flow, we may derive a very useful equation commonly known as Bernoulli's equation in honor of Daniel Bernoulli who proposed it. In the derivation of the equation, it is necessary to make certain assumptions which must be clearly understood in order to employ the equation intelligently. These assumptions and an explanation of what they involve may be summarized as follows:

(1) The fluid is frictionless. The assumption amounts to neglecting the effect of the tangential forces shown in Fig. 48 so that only normal pressures, in addition to the inertia forces, are considered to act on any plane within the fluid. Since the viscosity of fluids varies, the magnitude of the error produced by this assumption will depend upon the viscosity of the particular fluid in question, and upon the value of the last factor in Eq. (26), namely, dV/dy, the rate of change of the velocity. Near the solid walls, this ratio, or "gradient," is very large and Bernoulli's Law is not applicable for such conditions.

(2) The flow is steady. This assumption is made in order to exclude those types of flow wherein changes of velocity at any one section in the fluid occur with time. Although this condition in its strictest interpretation is purely ideal since minute fluctuations of velocity exist even in the smoothest flow, many cases of actual flow exist in which the average velocity over considerable periods of time remains constant. Such a flow exists at any point in a pipe discharging a constant quantity of fluid. Actual measurements show instantaneous fluctuations of velocity at points in the cross section of the pipe, but these fluctuations do not differ very much from the average velocity as determined over relatively long periods of time.

(3) The flow is through a stream tube. This limitation is imposed because in a stream tube the velocity of all particles at any one cross section is the same. In streams of larger cross section the velocities of particles generally vary from point to point in the cross section.

(4) The fluid is incompressible. This assumption is very nearly approached by liquids and can be adopted even for gases under certain conditions in which pressure changes are small.

(5) There is continuity of flow, so that the same quantity of fluid passes any cross section during the same interval of time.

Figure 55 shows a portion of a stream tube in which a fluid of specific weight, w, is flowing, included between the cross-sectional areas A_1 and A_2. At (1) the intensity of pressure in pounds per square foot is p_1, and at (2) it is p_2. The pressures on the sides of the stream tube are not shown because

they act normally and have no displacement as the mass of fluid moves along the tubes. As will be apparent presently, we are interested only in those forces which do work on the mass of fluid. The pressures on the sides of the tube do no work. The elevation of the stream tube at (1) is Z_1 and at (2) is Z_2. The velocities at the corresponding points are V_1 and V_2.

In a short interval of time, dt, the mass of fluid being considered will be displaced a short distance along the tube so that A_1 will move a small distance, ds_1, to A_1', and A_2 will move a small distance, ds_2, to A_2'.

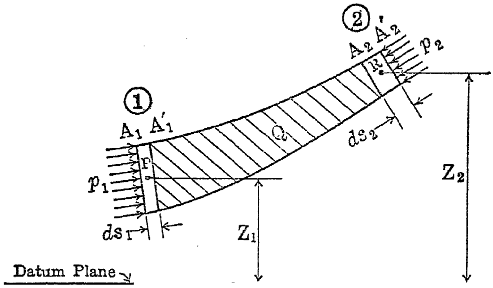


FIG. 55

Let us apply the principle of work and change in energy to the forces acting upon the mass of fluid. The principle states that the resultant work of the action of the external forces upon a given mass is equal to the change in kinetic and potential energies of the mass. The external forces doing work are those acting on A_1 and A_2. The work done by the force acting on A_1 is $p_1A_1ds_1$, and is positive because the force and displacement are in the same direction. For A_2, the work done is $p_2A_2ds_2$, and is negative because the force and displacement are in opposite directions. The net work done by the pressure forces during the interval of time is, therefore,

$$p_1A_1ds_1 - p_2A_2ds_2 \tag{a}$$

This work may be equated to the change in the kinetic and potential energies of the mass.

At the beginning of the interval of time the mass consists of the portions marked P and Q in Fig. 55; at the end it is made up of the portions Q and R. The portion, Q, is common to both, and since the flow is steady, the energy of Q is constant. The change in energy, consequently, is merely the difference between the energies of R and P. Considering potential energy first, the potential energy of R is $wA_2ds_2Z_2$; that of P is $wA_1ds_1Z_1$. The change in potential energy is the difference between these, or

$$wA_2ds_2Z_2 - wA_1ds_1Z_1 \tag{b}$$

Since kinetic energy is $\frac{1}{2}MV^2$, the kinetic energy of R is $\frac{1}{2}(wA_2ds_2/g)V_2^2$ and for P is $\frac{1}{2}(wA_1ds_1V_1^2/g)$. Their difference is

$$\frac{wA_2ds_2V_2^2}{2g} - \frac{wA_1ds_1V_1^2}{2g} \tag{c}$$

Combining (a), (b), and (c) into the principle of work and energy, we obtain

$$p_1A_1ds_1 - p_2A_2ds_2 = wA_2ds_2Z_2 - wA_1ds_1Z_1 + \frac{wA_2ds_2V_2^2}{2g} - \frac{wA_1ds_1V_1^2}{2g} \tag{38}$$

From continuity of flow

$$wA_1ds_1 = wA_2ds_2$$

so that we may divide each term of Eq. (38) by either wA_1ds_1 or wA_2ds_2, whichever is more convenient. Doing this, there results

$$\frac{p_1}{w} - \frac{p_2}{w} = Z_2 - Z_1 + \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

or

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 = C, \text{ a constant} \tag{39}$$

along any one streamline. This is the well-known Bernoulli equation for steady flow and is the basis for many of the equations of fluid motion. The equation in this form may be used only if the ideal conditions assumed in its derivation are closely simulated.

41. Significance of Terms in Bernoulli's Equation. Concept of Energy. Head.—A closer examination of the Bernoulli equation will bring out the fact that each of its terms represents a kind of energy and that in the form of Eq. (39) it is a statement of the conservation of mechanical energy when friction is neglected. The term, Z, is the elevation in feet of the point in the stream tube above a datum plane. As such it is numerically equal to the potential energy of each pound of the fluid as it passes the point in foot pounds per pound. The $V^2/2g$ is also recognizable as the kinetic energy of 1 lb. of fluid and can be expressed in foot pounds per pound or merely as a height in feet. Although p/w is a term which also may be represented by a height, its real significance requires a closer analysis. If we go back to the derivation of the Bernoulli equation in the preceding article, we find that this term is obtained by considering the work done by the pressures on the mass of fluid between the areas A_1 and A_2. Confining our attention to A_1, if we imagined this area to be replaced by a small piston of the same area, Fig. 56, the pressure against the piston could do an amount of work equal to $p_1A_1ds_1$ in moving it a distance ds_1. The weight of fluid passing A_1 in doing this work would be wA_1ds_1, so that the work per pound of fluid

which could be done by the fluid by virtue of its pressure is

$$\frac{p_1 A_1 ds_1}{w A_1 ds_1} = \frac{p_1}{w}\frac{\text{ft.lb.}}{\text{lb.}} = \frac{p_1}{w}\text{ ft.}$$

Merely because no piston like the one described above exists does not alter the fact that the fluid possesses energy by virtue of its pressure or simply pressure energy. Whether this energy is utilized in actually doing work against a piston or in altering the potential and kinetic energy of a mass of fluid as assumed in the derivation of the Bernoulli equation is of no consequence.

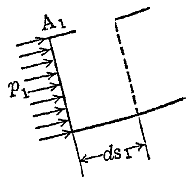


FIG. 56

Since all three of the above forms of energy are expressible in feet, they are generally referred to as heads. For example, p/w is called pressure head; $V^2/2g$, velocity head; and Z, elevation head. The sum of these at any point is the total energy or total head. If we represent the sum of these at any point in the fluid by H, then

$$H = \frac{p}{w} + \frac{V^2}{2g} + Z \tag{40}$$

in feet of fluid flowing.

The Bernoulli equation, therefore, is merely a statement of the law of conservation of energy, namely, that the total energy of the fluid at one point in a streamline is equal to the total energy at some other point along the same streamline, provided that there are no losses of energy between the two points.

The student will recall that one of the assumptions made in the derivation of the Bernoulli equation was the absence of tangential or frictional forces. The action of these frictional forces results in a loss of energy similar to that occurring due to friction between solid bodies. The exact nature and amount of this frictional energy loss depend upon many factors, and cannot be determined analytically except in a few isolated cases. However, since it is an energy loss, and since in many instances it is relatively large, it is included in the Bernoulli equation by adding a term to the equation. The Bernoulli equation between two points, (1) and (2), along the streamline then becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 + h_f \tag{41}$$

in which, h_f, is the term representing the energy lost between points (1) and (2) due to friction in feet of liquid flowing, and is called the friction head. Methods for determining h_f in problems involving pipe flow are

given in Chap. VIII. In the problems of this chapter, h_f will either be given, or will be the only unknown in the equation so that it may be computed.

Equation (41) is the more general equation for the flow of liquids and takes the form of Eq. (39) when the energy loss, h_f, is small enough to be neglected without appreciable error.

42. Bernoulli Equation for Actual Streams. — The Bernoulli principle as embodied in Eq. (41) applies strictly to a single streamline. In many problems of practical importance, however, such as in pipe flow, the stream is made up of a group of streamlines or stream tubes. In general, the velocity varies from point to point in a cross section of the stream, and usually only the average velocity as determined by dividing the total discharge, Q, by the area, A, is obtainable. Even under these conditions the Bernoulli equation may be used without serious error, due to the fact that the velocity head term is generally small in comparison to the other terms. This will be demonstrated in the following paragraphs.

The Bernoulli equation as derived in Art. 40 states that the total energy of the fluid passing a cross section of a stream tube is equal to the total energy at some other section plus the losses occurring between the two sections. It follows from this that the total energy of a stream made up of stream tubes, at one cross section, is equal to the total energy at some other cross section plus the losses occurring between the two sections.

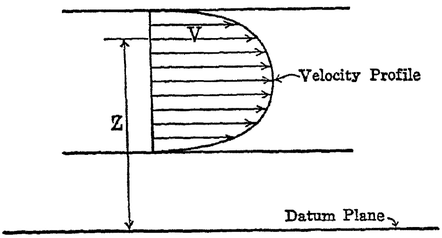


Fig. 57

Since the quantity flowing past each cross section in steady flow during the same time interval is the same, the average energy per pound of fluid at one section is equal to the average at the second section plus the loss between the two sections. This average energy per pound, or head, depends upon the velocity distribution in the cross section and may be found in the following manner.

Figure 57 shows a cross section of a stream with an assumed velocity variation as represented by the velocity profile. Let us determine the total and average energy at the section. The total head at any point in the cross section is

$$H = \frac{p}{w} + \frac{V^2}{2g} + Z$$

in foot pounds per pound. The weight of fluid passing an elementary area,

dA, in an interval of time, t, is

$$W = wVdA \cdot t$$

in pounds, so that the energy passing the elementary area is

$$dH = \left[\frac{p}{w} + \frac{V^2}{2g} + Z\right]wVdA \cdot t$$

$$= \left[\frac{p}{w} + Z\right]wVdA \cdot t + \frac{wV^3dA \cdot t}{2g}$$

The total energy passing the section in foot pounds is then

$$H = \int\left[\left(\frac{p}{w} + Z\right)wVdA \cdot t + \frac{wV^3dA \cdot t}{2g}\right] \tag{42}$$

For a stream with parallel, rectilinear flow the quantity, $p/w + Z$, may be shown to be a constant for every point in the cross section, as follows. Figure 58 shows an elementary vertical prism of fluid in a horizontal stream. The forces acting on the prism in a vertical direction are the pressures at the top and bottom, and the weight. Since the mass of fluid being considered is undergoing no acceleration in the vertical direction, the summation of vertical forces must equal zero or

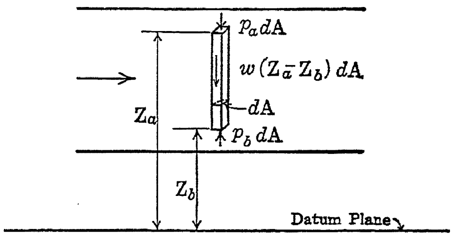


FIG. 58

$$p_a dA + w(Z_a - Z_b)dA - p_b dA = 0$$

$$\frac{p_a}{w} + Z_a = \frac{p_b}{w} + Z_b = C, \text{ a constant} \tag{43}$$

for any point in the cross section. Since the sum of the elevation head and pressure head is the same for any point in the cross section, it matters not what point in the area is chosen for its computation. In pipe flow, the pressure head and elevation head at the centerline of the pipe are used.

Equation (42), therefore, can be written as

$$H = \left(\frac{p}{w} + Z\right)\int wVdA \cdot t + \int \frac{wV^3dA \cdot t}{2g}$$

Since $\int wVdA \cdot t$ is the weight flowing past the cross section during the time interval, t, the average energy per pound of fluid is then the total

energy divided by the weight, or

$$H_{ave} = \frac{\left(\frac{p}{w} + Z\right)\int wVdA \cdot t + \int \frac{wV^3dA \cdot t}{2g}}{\int wVdA \cdot t}$$

$$= \frac{p}{w} + Z + \frac{1}{2g}\frac{\int V^3dA}{\int VdA} \tag{44}$$

The last term in this expression can be evaluated exactly only when the velocity variation in terms of dA is known, or by graphical integration of the numerator. As stated in the first paragraph of this article, it is not the velocity variation, but the average velocity, that is known. If we use this average velocity in computing the average kinetic energy of the fluid per pound of fluid flowing, we introduce an error equal to the difference between $(1/2g)(\int V^3dA/\int VdA)$ and $\bar{V}^2/2g$, in which $\bar{V}$ represents the average velocity. The student should note that $\int VdA$ is merely the discharge, and is equal to $A\bar{V}$.

The true average kinetic energy head, $(1/2g)(\int V^3dA/A\bar{V})$, is always larger than the kinetic energy head as computed from $\bar{V}^2/2g$ so that the former may always be expressed as $\alpha(\bar{V}^2/2g)$, where α has a value varying usually between 1 and 2. α is 1 when the velocity is constant over the cross section and 2 when the velocity varies parabolically from zero at the wall of a pipe to a maximum at the center. These values represent extremes in pipe flow, and since the kinetic energy is only a portion of the total energy, no large error is involved if the centerline is taken as a streamline with a velocity equal to the average velocity. Thus the equation,

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 + h_f \tag{41}$$

is applicable even for pipe flow where the subscripts, 1 and 2, represent points along the centerline and the velocities are the average velocities at the two cross sections.

In other types of flow, such as that over weirs, the above approximation for the average kinetic energy may not be permissible without causing relatively large errors, so that often it is necessary to obtain a closer value

for the average kinetic energy of the stream by using the factor α mentioned above.

Illustrative Problem. In an open channel of depth, d, and width, b, the velocity varies from a maximum at the surface to zero at the bottom according to the equation

$$V_y = V_{\text{max.}}\left[1 - \left(\frac{y}{d}\right)^2\right]$$

in which y is the depth to the velocity, V_y. Figure 59 shows the velocity profile for these conditions. Assuming no velocity variation in a direction perpendicular to the plane of the paper, find

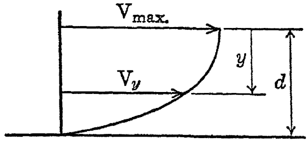


FIG. 59

(*a*) the average velocity in terms of $V_{\text{max.}}$

(*b*) the factor α.

(*a*) The discharge through an elementary slit, b wide and dy high, at a depth y is

$$dQ = V_y dA = V_y b dy$$

$$V_y = V_{\text{max.}}\left[1 - \left(\frac{y}{d}\right)^2\right]$$

Therefore

$$dQ = V_{\text{max.}}\left[1 - \left(\frac{y}{d}\right)^2\right] b dy$$

and

$$Q = bV_{\text{max.}}\int_0^d \left[1 - \left(\frac{y}{d}\right)^2\right] dy$$

$$= \tfrac{2}{3} bdV_{\text{max.}}$$

The average velocity is

$$\bar{V} = \frac{Q}{A} = \frac{\frac{2}{3}bdV_{\text{max.}}}{bd} = \tfrac{2}{3}V_{\text{max.}} \qquad \textit{Ans.}$$

(*b*) The true kinetic energy per pound of fluid flowing was shown to be

$$\frac{1}{2g}\frac{\int V^3 dA}{A\bar{V}} = \frac{V^3_{\text{max.}}\, b \int \left[1 - \left(\frac{y}{d}\right)^2\right]^3 dy}{2g \cdot \frac{2}{3} bdV_{\text{max.}}}$$

Expanding the term in the parenthesis, this becomes

$$\frac{3}{2}\frac{V^2_{\text{max.}}}{2gd}\int_0^d \left[1 - 3\left(\frac{y}{d}\right)^2 + 3\left(\frac{y}{d}\right)^4 - \left(\frac{y}{d}\right)^6\right] dy = \frac{48}{70}\frac{V^2_{\text{max.}}}{2g}$$

The kinetic energy per pound of fluid based upon the average velocity, $\bar{V}$, is

$$\frac{\bar{V}^2}{2g} = \frac{(\frac{2}{3}V_{\text{max.}})^2}{2g} = \frac{4}{9}\frac{V^2_{\text{max.}}}{2g}$$

Therefore

$$\frac{48}{70}\frac{V^2_{\text{max.}}}{2g} = \alpha\frac{\bar{V}^2}{2g} = \alpha\frac{4}{9}\frac{V^2_{\text{max.}}}{2g}$$

and

$$\alpha = \frac{70}{A} \doteq \qquad 1.54 \text{ ans.}$$

PROBLEMS

58. The velocity in a circular pipe varies parabolically from a maximum at the center to zero at the wall according to the law

$$V_r = V_c\left[1 - \left(\frac{r}{r_o}\right)^2\right]$$

In this equation V_r is the velocity at radius r; V_c is the velocity at the center; and r_o is the radius of the pipe. Find:

(*a*) the average velocity in terms of V_c.

(*b*) the value of α.

Hint: Use an annular ring dr wide as the elemental area.

59. A liquid flows between two parallel plates as shown in Fig. 60. The velocity varies according to the equation

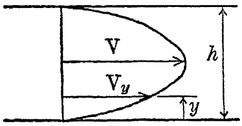


FIG. 60

$$V_y = \frac{4V}{h^2}(hy - y^2)$$

in which V_y is the velocity at a distance y from the bottom plate and V is the maximum velocity.

(*a*) Find the average velocity.

(*b*) Determine α.

60. Water is flowing in a 6 in. pipe at the rate of 5 c.f.s. At one point in the line, the pressure is 40 lb. per sq. in. Find the total head at the above point with reference to a datum plane which is 10 ft. below the pipe.

61. Gasoline, S.G. 0.72, flows in the pipe described in Prob. 60, while all other data remains the same. Find the total head in feet of gasoline at the point.

43. Applications of Bernoulli's Equation. — The basic equation used in the solution of most problems involving the flow of liquids is the Bernoulli equation or

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 + h_f$$

Although it involves seven terms, the two velocity head terms are not independent because either velocity may be obtained in terms of the other by the use of the continuity equation,

$$Q = A_1V_1 = A_2V_2$$

Thus the equation may be said to include six terms. Further simplification can often be obtained due to the fact that h_f may be expressed as a function of the velocity head. Usually the equation must be written between two points on the same streamline for which all but one of the remaining quantities are known. The unknown quantity can then be calculated.

Sometimes it is necessary to apply Bernoulli's equation to two points, one of which may not be of primary interest, in order to obtain information

which may be used for a second application of the equation. An example of this is given in the second of the following illustrative problems.

Illustrative Problem 1. The diameter of a horizontal pipe through which water flows changes from 6 in. to 12 in. The intensity of pressure at the 6 in. point is 10 lb. per sq. in. and that at the 12 in. point is 20 lb. per sq. in. Find the direction of flow and the frictional loss when the pipe carries 5 c.f.s.

All the terms in the Bernoulli equation can be evaluated directly with the given data except h_f. Thus,

$$\frac{p_1}{w} = 23.08 \text{ ft.}; \quad \frac{p_2}{w} = 46.16 \text{ ft.}$$

$$\frac{V_1^2}{2g} = \frac{(5/0.196)^2}{2g} = 10.1 \text{ ft.}$$

$$\frac{V_2^2}{2g} = \frac{(5/0.7854)^2}{2g} = 0.63 \text{ ft.}$$

If the datum plane is taken through the center of the pipe, the total head at the 6 in. point is

$$23.08 + 10.1 + 0 = 33.18 \text{ ft.}$$

and the total head at the second point is equal to

$$46.16 + 0.63 + 0 = 46.79 \text{ ft.}$$

The total head at the 12 in. end is greater than that at the 6 in. end, therefore the flow is from the 12 in. end towards the 6 in. end. The frictional loss is the difference between the total heads at the two ends, which is

$$46.79 - 33.18 = 13.61 \text{ ft.}$$

Illustrative Problem 2. Water flows from the reservoir through the pipeline shown in Fig. 61. Assuming the head loss between A and B to be 3 ft. and that between B and C to be 2 ft., find the pressure head at B. The reservoir is so large that the velocity at A is negligible.

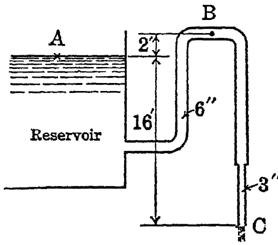


Fig. 61

The datum plane will be chosen through the lowest point C so that the elevation head for the three points entering in the problem will be positive.

If we attempt to solve directly for the pressure head at B by writing Bernoulli's equation between A and B we have

$$\frac{p_A}{w} + \frac{V_A^2}{2g} + Z_A = \frac{p_B}{w} + \frac{V_B^2}{2g} + Z_B + h_f \qquad \text{(a)}$$

In this equation p_A/w is atmospheric or zero; $V_A^2/2g$ is zero; Z_A is 16 ft.; p_B/w and $V_B^2/2g$ are unknown; Z_B is 18 ft.; and h_f is 3 ft. Since we have but one equation and two unknowns it is impossible to solve directly.

However, if the equation is written between points A and C, this difficulty can be overcome. The stream at C is subjected only to the atmosphere so p_C/w is

zero. h_f between A and C is equal to 5 ft. Substituting in Bernoulli's equation,

$$0 + 0 + 16 = 0 + \frac{V_C^2}{2g} + 0 + 5$$

$$\frac{V_C^2}{2g} = 11 \text{ ft.}$$

From the equation of continuity

$$A_B V_B = A_C V_C$$

and

$$d_B^2 V_B = d_C^2 V_C$$

in which d_B and d_C are the diameters at B and C respectively. Therefore

$$V_B = \left(\frac{d_C}{d_B}\right)^2 V_C$$

and

$$\frac{V_B^2}{2g} = \left(\frac{d_C}{d_B}\right)^4 \frac{V_C^2}{2g} = \left(\frac{3}{6}\right)^4 \frac{V_C^2}{2g} = \frac{11}{16} = 0.69 \text{ ft.}$$

Inserting this value for $V_B^2/2g$ in (a),

$$0 + 0 + 16 = \frac{p_B}{w} + 0.69 + 18 + 3$$

and

$$\frac{p_B}{w} = -5.69 \text{ ft.}$$

The negative sign indicates that the pressure is less than atmospheric, or that a partial vacuum exists at point B.

Two important points are brought out in the solution of the above problem. First, the student should note that the *velocity heads* at C and B are determined, and not the velocities. There is no advantage in solving for the velocities in this problem because it would be necessary to convert these velocities back to velocity heads in order to complete the solution. Secondly, although the velocities vary inversely as the square of the diameters, the velocity heads vary inversely as the *fourth* power of the diameters. These warnings may seem trivial, but it has been the experience of the authors that students make many errors because they fail to take cognizance of these two points.

PROBLEMS

62. Water flows down through the draft tube of Fig. 62 at the rate of 250 c.f.s. Considering no losses, find the pressure head at A in feet of water.

63. A pipe expands from 6 in. in diameter at A to 12 in. in diameter at B. Point A is 10 ft. below point B. The pressure at A is 10 lb. per sq. in. and that at B is 15 lb. per sq. in. The pipe carries 10 c.f.s. of oil (S.G. = 0.8). Find the lost head and the direction of flow. *Ans.* $h_f = 13.46$ ft.

64. A pipe expands from 6 in. in diameter at A to 12 in. in diameter at B. Points A and B are at the same elevation and the pressure at B is 10 lb. per sq. in. greater than that at A. The pipe carries 8 c.f.s. of water. Find the direction of flow and the lost head.

65. The pipe described in Prob. 64 carries 6 c.f.s. Find the direction of flow and the lost head.

66. A jet of water which is directed upward is 8 in. in diameter and has a velocity of 50 ft. per sec. Find the velocity of the water and the diameter of the jet at a point 25 ft. higher.

67. Water is flowing through the tube shown in Fig. 63. A mercury manometer is connected to the tube as indicated and shows a deflection of 10 in.

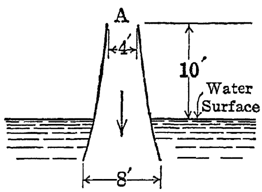


FIG. 62

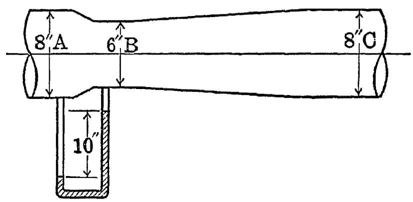


FIG. 63

(*a*) Neglecting friction between A and B, find the discharge.

(*b*) Assuming the friction loss between B and C to be 0.15 h, where h is the difference between the pressure heads at A and B, find the pressure head at C.

68. What would be the discharge, and pressure at C, for the conditions of Prob. 67, if oil with a specific gravity of 0.7 is flowing through the tube?

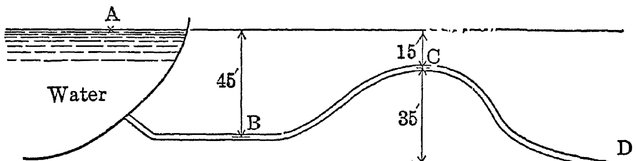


FIG. 64

69. The loss of head for the set-up shown in Fig. 64 from A to B is one velocity head, from B to C is one velocity head, and from C to D is two velocity heads. Find the pressure head in feet of water at B and C if the pipe is 6 in. in diameter.

70. Suppose that all of the data in Prob. 69 remains the same except the diameter at C. Find this diameter if there is a vacuum of 20 in. of Hg. at C.

Ans. $d_c = 5.2$ in.

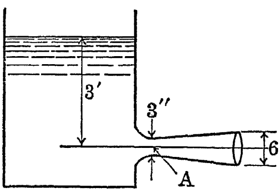


FIG. 65

71. Suppose that the diameter at C in Prob. 69 remains 6 in. and all other data are likewise unchanged except the elevation of C. How far above D can C be placed to produce a vacuum of 20 in. of Hg.?

72. Find the pressure at A in Fig. 65. Assume the flow to be frictionless. For what fluid is this possible?

73. To what height above the center of the diverging tube must the water in the container of Fig. 65 be raised in order to cause a pressure at A equal to a vapor pressure of water of 0.5 lb. per sq. in. abs. Assume atmospheric pressure as 14.7 lb. per sq. in. and neglect friction. What would happen if the water surface were raised above this elevation?

44. Flow in Curved Paths. — The Bernoulli equation as derived for a frictionless fluid stated that the total head or energy *along* the same streamline was equal to a constant. Also, in Art. 42, it was stated that in a stream made up of parallel, rectilinear streamlines $p/w + Z$ was a constant. If, in addition, there is no velocity variation in a cross section, the velocity head, $V^2/2g$, is the same for each point in the cross section. Under these assumed conditions of steady flow, there is no variation of the Bernoulli constant either along a streamline or as one moves from streamline to streamline. Thus, in this kind of flow the total head is the same throughout the fluid.

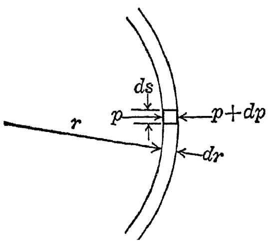


Fig. 66

It remains now to investigate conditions in which the flow is not rectilinear, but where the streamlines possess appreciable curvature. Figure 66 shows an element of fluid moving in a curved path in a horizontal plane. The radius of curvature of the path is r; the dimensions of the particle in the plane of the paper are dr and ds, and the length perpendicular to the plane of the paper is unity. Since the particle is undergoing an acceleration, V^2/r, towards the center of curvature, there must be a resultant force in this direction equal to the mass of the particle multiplied by its acceleration. Therefore, the intensity of pressure on the outside face of the particle must be greater than that on the inside. We indicate these two pressures by p and $p + dp$, respectively. The resultant force towards the center is

$$(p + dp)\, ds - pds \tag{a}$$

The volume of the particle is $drds$ and, if the specific weight of the fluid is w, its mass is

$$\frac{w}{g}\, drds \tag{b}$$

Therefore

$$(p + dp)\, ds - pds = \frac{w}{g}\, drds\, \frac{V^2}{r}$$

and
$$dp = \frac{w}{g}\frac{V^2}{r}\,dr \tag{45}$$

In order to integrate Eq. (45), it is necessary to express V in terms of r. There are two types of flow, approximated in practice, for which this relationship is known. In the first of these, the velocity along the path varies directly as the distance from the center of curvature of the path (Fig. 67). In this case, calling ω the angular velocity of a point about a vertical axis through the center of curvature, we have

$$V = \omega r \tag{a}$$

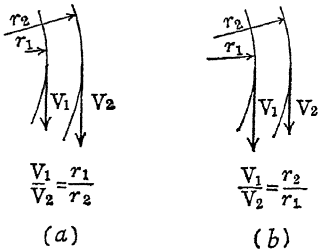


FIG. 67

Substituting in Eq. (45), above,

$$dp = \frac{w}{g}\frac{\omega^2 r^2}{r}\,dr$$

Integrating between any two points, (1) and (2), along a radius, there results

$$p_2 - p_1 = \frac{w\omega^2}{2g}(r_2^2 - r_1^2) \tag{46}$$

or
$$\frac{p_1}{w} - \frac{V_1^2}{2g} = \frac{p_2}{w} - \frac{V_2^2}{2g} \tag{47}$$

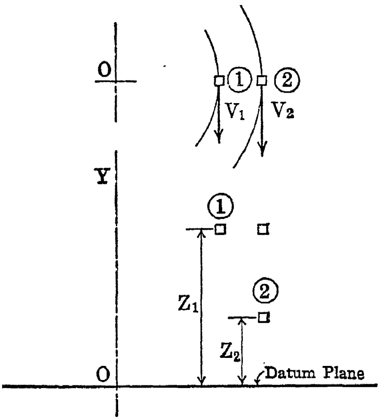


FIG. 68

Equations (46) and (47) apply between two points in the same horizontal plane, or between any two points for fluids in which differences in pressure produced by differences in elevation are negligible. For two points not at the same elevation rotating in concentric circles about a vertical axis, the equation must be written as

$$\frac{p_1}{w} - \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} - \frac{V_2^2}{2g} + Z_2 \tag{48}$$

The truth of this statement may be verified by a study of Fig. 68. Since Eq. (48) differs from the Bernoulli equation, the Bernoulli equation *cannot* be applied between points along a radius when the velocity is known to vary according to (a), but the relationship between the heads is given by Eq. (48).

In the second type of flow mentioned above, the velocity varies inversely

as the distance from the center of curvature, or

$$\frac{V_1}{r_2} = \frac{V_2}{r_1} \tag{b}$$

Thus, if the velocity, V_1, at a radius, r_1, is known, the velocity, V, at any other point along the radius is

$$V = \frac{r_1 V_1}{r} \tag{c}$$

Substituting this value for V in Eq. (45), we obtain

$$dp \quad \frac{w}{g} r_1^2 V_1^2 \frac{dr}{r^3}$$

Integrating between two points, (1) and (2), along the radius

$$p_1 - p_2 = \frac{w}{2g} r_1^2 V_1^2 \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) \tag{49}$$

Since, from (c),

$$\frac{V_1^2}{r_2^2} r_1^2 = V_2^2$$

$$\frac{p_1}{w} - \frac{p_2}{w} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

and

$$\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} \tag{50}$$

This is the Bernoulli equation for two points at the same elevation, or for any two points in fluids of small specific weights in which differences in elevation are negligible. For two particles at different elevations rotating about a vertical axis according to (b), the equation becomes

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2$$

which is the Bernoulli equation. Therefore, for this type of steady motion the Bernoulli equation gives the relationship between heads at any two points in the fluid and not only along a streamline.

A clearer understanding of the conditions under which the two types of motion discussed above may be expected can be had by studying the deformation of particles resulting from the assumed velocity variations.

Thus in Fig. 69*a* is shown a particle in a fluid within which the velocity variation is given by the equation

$$V = \omega r$$

The particle in moving a short distance along its path takes the positions indicated by (1) and (2) in the figure. In so doing the side *rs* moves relative to *mn*. For this to be possible a shearing stress on the face *rs* must be applied and a bodily rotation of the particle results. For this reason, this type of flow is called a *rotational* flow and may be expected where high relative velocities between particles exist or where relatively large viscous forces come into play. This is almost always the case in the region next to a solid boundary.

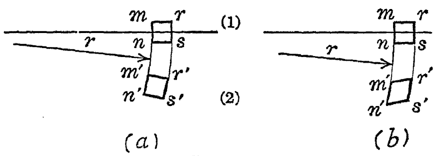


FIG. 69

Figure 69*b* shows a particle in a fluid moving so that the velocity variation is given by the equation

$$\frac{V_1}{r_2} = \frac{V_2}{r_1}$$

In this case, for a small displacement of the particle, the sides *mn* and *rs* pass through distances inversely proportional to their radii and since we are considering a fluid with no viscosity, there is no shearing force on the face *rs*. Although the particle changes its shape as shown, there is no bodily rotation of the particle and the motion is said to be *irrotational.* Since an irrotational flow implies the absence of shearing forces, this type of flow is possible strictly only in a perfect fluid. However, it occurs approximately with fluids of low viscosity in regions where the relative velocity between particles is small. Thus, this type of flow is not likely near solid boundaries where the large relative velocities between particles give rise to high viscous forces. This region of large relative velocity is usually confined to a thin boundary layer, however, outside of which the flow is practically irrotational.

The important conclusions to be remembered concerning steady curvilinear motion are:

(1) In general, the Bernoulli equation can be applied only *along* a streamline.

(2) The Bernoulli equation can be applied throughout a region where the flow is known to be irrotational.

(3) Irrotational flow may be expected in regions outside the immediate vicinity of solid boundaries where large differences in the velocity of adjacent particles do not exist.

PROBLEMS

74. The cylindrical container shown in Fig. 70 is 4 ft. in diameter. It contains water and rotates about the axis OY with an angular velocity of 25 r.p.m. Because of viscosity, each particle of fluid has the same angular velocity as the cylinder. Centrifugal force causes the surface of the water to assume the shape indicated by AOB.

(*a*) What is the intensity of pressure at C, a point at the same elevation as O, 1 ft. from the axis of rotation?

(*b*) How high vertically above C is the surface of the liquid?

(*c*) Find the maximum theoretical elevation of B above O.

(*d*) What mathematical curve is AOB?

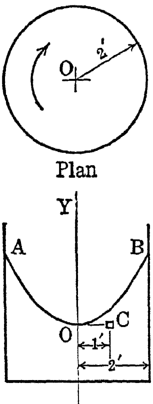


Fig. 70

75. The closed cylindrical container shown in Fig. 71 is 3 ft. in diameter and is filled with oil having a specific gravity of 0.85. By means of paddles, the oil is rotated so that each particle has an angular velocity of 100 r.p.m. about OY. To what height above O will the oil rise in the open tube?

Ans. $h = 3.82$ ft.

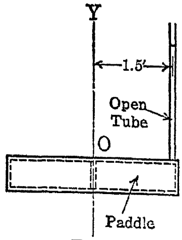


Fig. 71

76. Figure 72 shows the streamline diagram for air flowing past a streamlined body. At point A the distance between streamlines is 0.7 in., the velocity is 70 ft. per sec., and the pressure 14.7 lb. per sq. in. abs. At B, the streamlines are 0.3 in. apart. Assuming the flow to be irrotational and neglecting differences in elevation and compressibility, what is the pressure at B? Take the weight of air as 0.081 lb. per cu. ft.

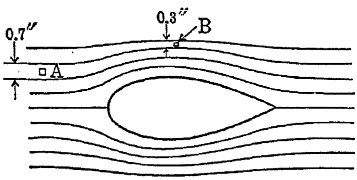


Fig. 72

77. A tornado may be considered as a body of air rotating about an axis as shown in Fig. 73. The flow is assumed to be irrotational except in the vicinity of the centerline. If at a distance of 200 ft. from the axis of rotation the velocity is 10 m.p.h. and the pressure 14.2 lb. per sq. in. abs., find the velocity and pressure at a distance of 15 ft. from the centerline. Take the average weight of air between the two points as 0.07 lb. per cu. ft.

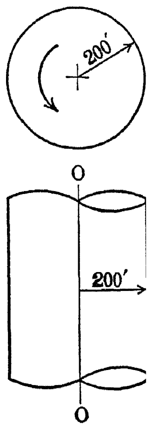


Fig. 73

45. Bernoulli Equation for Compressible Fluids.— The Bernoulli equation for compressible fluids is more complicated than that for liquids, because in dealing with gases or vapors the variation of specific weight due to pressure changes must be taken into account. This variation in specific weight is intimately connected with heat changes which arise

because of the interdependence of temperature, pressure, and volume. Since heat is thus seen to be an important factor in the consideration of compressible fluids, any energy relationships for these fluids must include heat as well as the other forms of energy.

In Chap. I, it was stated that a gas possessed intrinsic energy because of the motion of its molecules, and that it was possible for a gas to do work at the expense of this intrinsic or heat energy. It was further stated that heat energy might be converted into mechanical energy at the rate of 778 ft. lb. per B.t.u. In the following discussion, this factor for the mechanical equivalent of heat will be indicated by the letter J. With the foregoing as groundwork, we proceed to the derivation of the equation for compressible fluids in much the same manner employed in deriving the Bernoulli equation for liquids.

The same assumptions that were made in Art. 40 for liquids will be adopted here with the exception that the specific weight, w, is variable. Since the flow is still assumed to be steady, the equation of continuity states that the weight flow past each section per unit of time is the same or

$$W = w_1 A_1 V_1 = w_2 A_2 V_2,$$

in which W is the weight flow in pounds per second; w is the specific weight in pounds per cubic foot; A is the area in square feet; and V is the velocity in feet per second.

Figure 74 shows a short length Δs, of a stream tube through which a compressible fluid is flowing. In the length, Δs, the area changes from A at the point marked (1) to $(A + dA)$ at (2). In a short interval of time the mass of fluid being considered moves along the tube and the areas A and $(A + dA)$ move small distances ds_1 and ds_2 respectively. At point (1) the pressure is p, the velocity is V, the elevation is Z, and the intrinsic energy in B.t.u.'s per pound is I. At point (2) these terms have all changed by the small amounts indicated underneath the figure. In order to make the treatment general, an amount of heat energy, dQ, in B.t.u. per pound of fluid flowing, is assumed to be supplied to the fluid between the two sections.

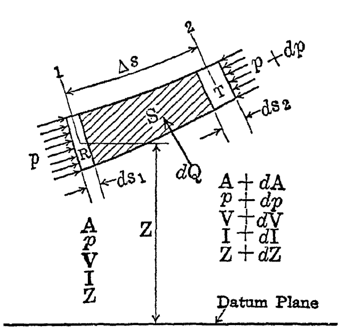


FIG. 74

The principle of work and energy for these conditions is that the work done by the external forces plus the external energy added is equal to the

total change in energy. This change is the difference between the final and initial sums of potential, kinetic, and heat energies.

The work done by p is $pAds_1$ and is positive; that done by $(p + dp)$ is $(p + dp)(A + dA)ds_2$ and is negative. The net amount of work done by these two external forces during the interval is therefore

$$pAds_1 - (p + dp)(A + dA)ds_2 \quad \text{ft. lb.} \tag{a}$$

During the interval of time, the weight flow past any section is $wAds_1$ in pounds so that, in addition, energy equal to $wAds_1dQJ$ is supplied to the mass from an outside source. The total energy supplied to the mass is equal to

$$pAds_1 - (p + dp)(A + dA)ds_2 + wAds_1dQJ \quad \text{ft. lb.} \tag{b}$$

This external energy is utilized in changing the total energy of the given mass. Since steady flow was assumed, the energy of the portion marked S remains constant so that the change in energy is the difference between the energies of T and R. Considering potential energy first, the potential energy of T is $(w + dw)(A + dA)ds_2(Z + dZ)$ and that of R is $wAds_1Z$. Their difference is

$$(w + dw)(A + dA)ds_2(Z + dZ) - wAds_1Z \quad \text{ft. lb.} \tag{c}$$

Similarly the change in kinetic energy is

$$\frac{(w + dw)(A + dA)ds_2}{2g}(V + dV)^2 - \frac{wAds_1V^2}{2g} \quad \text{ft. lb.} \tag{d}$$

and the change in intrinsic energy is

$$(w + dw)(A + dA)ds_2(I + dI)J - wAds_1IJ \quad \text{ft. lb.} \tag{e}$$

From continuity of flow,

$$wAds_1 = (w + dw)(A + dA)ds_2$$

so that each of the above energy terms may be divided by either $wAds_1$ or $(w + dw)(A + dA)ds_2$. Doing this and combining the resulting terms into the statement of the principle of work, there results

$$\frac{p}{w} - \frac{p + dp}{w + dw} + JdQ = dZ + \frac{V^2 + 2VdV + dV^2 - V^2}{2g} + JdI$$

or, collecting the first and second terms over a common denominator,

$$\frac{pw + pdw - pw - wdp}{w^2 + wdw} + JdQ = dZ + \frac{2VdV - dV^2}{2g} + JdI$$

Now, since wdw can be neglected in comparison with w^2, and $(dV)^2$ is a differential of the second order and negligible in comparison to $2VdV$, the

equation reduces to

$$\frac{pdw - wdp}{w^2} + JdQ = dZ + \frac{VdV}{g} + JdI$$

But

$$\frac{pdw - wdp}{w^2} = -d\left(\frac{p}{w}\right) = -d(pv)$$

in which v is the specific volume in cubic feet per pound. Then

$$JdQ = d(pv) + dZ + \frac{VdV}{g} + JdI \qquad \text{(I)}$$

$$= vdp + dZ + \frac{VdV}{g} + JdI + pdv \qquad (51)$$

Equation (51) is the differential equation for the flow of a compressible fluid. Upon integration between two points, (1) and (2), along a stream tube, it becomes

$$J_1Q_2 = p_2v_2 - p_1v_1 + Z_2 - Z_1 + \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + J(I_2 - I_1)$$

or

$$p_1v_1 + JI_1 + \frac{V_1^2}{2g} + Z_1 + J_1Q_2 = p_2v_2 + JI_2 + \frac{V_2^2}{2g} + Z_2 \qquad (52)$$

In this equation,

J = mechanical equivalent of heat which is 778 ft. lb. per B.t.u.,
p = absolute pressure of fluid in pounds per square foot,
v = specific volume in cubic feet per pound,
Z = vertical height above a datum plane in feet,
V = velocity of fluid in feet per second,
g = acceleration due to gravity, 32.2 ft. per sec.^{2},
I = internal energy of the fluid above that at some arbitrary temperature (usually 32° F.) in B.t.u. per pound,
${}_1Q_2$ = heat added to the fluid between the two points in B.t.u. per pound flowing.

Equation (52) is one form of the general equation for the flow of a compressible fluid. It merely states that the total energy, including heat, at one point along the flow plus the energy added between two points along the flow is equal to the total energy at the second point. Although the equation was derived on the basis of frictionless flow, it is applicable even to flow where friction is not negligible as is demonstrated in Art. 47.

In thermodynamics the quantity $I + pv/J$ for a substance is defined as

enthalpy or total heat, H. Equation (52) can therefore be written as

$$JH_1 + \frac{V_1^2}{2g} + Z_1 + J_1Q_2 = JH_2 + \frac{V_2^2}{2g} + Z_2. \tag{53}$$

The value of H has been determined experimentally for vapors of engineering importance, such as steam and ammonia, and can be obtained from tables or charts if the pressure, temperature and quality are known. Equation (53) is especially useful in dealing with vapors, but its full usefulness cannot be realized without a greater knowledge of thermodynamic principles than can be given in this text.

Another form of the energy equation for frictionless flow which is especially important in dealing with gases may be derived by considering the result of adding heat to a gas. This process was described in Art. 6, p. 5, for the case of a constant pressure change. There it was established that the heat supplied was utilized partly in increasing the intrinsic energy of the gas and partly in doing external work, or

$$J_1Q_2 = J(I_2 - I_1) + p(v_2 - v_1) \tag{L}$$

in which J = mechanical equivalent of heat in B.t.u.,
$_1Q_2$ = heat added in B.t.u. per pound,
$(I_2 - I_1)$ = change in intrinsic energy in B.t.u. per pound,
$p(v_2 - v_1)$ = external work done in foot pounds per pound.

However, if the heat is added while the pressure is caused to vary in any manner whatsoever, this equation must be written for a differential change in volume, dv, during which the pressure, p_1, may be considered constant. Thus

$$JdQ = JdI + pdv$$

Substituting this value of JdQ in Eq. (51) gives

$$0 = vdp + \frac{VdV}{g} + dZ$$

Integrating along a streamline,

$$\int_{p_1}^{p_2} vdp + \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + Z_2 - Z_1 = 0$$

or

$$\frac{V_1^2}{2g} + Z_1 = \frac{V_2^2}{2g} + Z_2 + \int_{p_1}^{p_2} vdp \tag{54}$$

Equation (54) is another form of the Bernoulli equation for compressible fluids neglecting friction, and can be used when v may be expressed as a function of p. For perfect gases the required relationship between pressure and volume at one point during expansion to the pressure and volume

at some other point can be expressed by the equation

$$pv^n = p_1v_1^n$$

in which p and v are the pressure and specific volume at any point, and p_1 and v_1 are a known pressure and corresponding specific volume at another point; n is an exponent depending upon the process. For isothermal flow n is 1, and for adiabatic flow n, the ratio of the specific heat at constant pressure to that at constant volume, is obtained from Table II, p. 9.

For an actual stream in which the streamlines are parallel and rectilinear, V may be taken as the average velocity, and Z, p, and v as the values for these terms at the center of the stream.

46. Applications of Bernoulli's Equation for Compressible Fluids without Friction. — Equations (53) and (54) are the basic equations for the flow of a compressible fluid. Equation (53) is especially applicable to the flow of vapors, but it usually requires a knowledge of temperature, pressure and quality at both points in the flow so that the enthalpy may be determined. Equation (54) finds its greatest use in the frictionless flow of gases, which more nearly obey the perfect gas laws. Its use depends upon a knowledge of the process so that $\int vdp$ may be evaluated. In both types of problems it is usually necessary to employ the equation of continuity for compressible fluids, i.e.,

$$W = w_1A_1V_1 = w_2A_2V_2$$

Examples of the application of the equations are given in the following illustrative problems.

Illustrative Problem 1. In running a test on a steam line the following data were obtained for two points along the pipe.

Diameter of pipe 6 in.	*Pipe horizontal.*
$p_1 = 120$ lb. per sq. in. abs.	$p_2 = 80$ lb. per sq. in. abs.
$T_1 = 341.3°$ F.	$T_2 = 317.0°$ F.

Steam at point (1) is dry saturated or has a quality of 1.
The pipe was very well insulated so that no heat was assumed to enter or leave the pipe.

Find the quantity of steam flowing in pounds per second.

From steam tables and the above data the following information may be obtained.

$v_1 = 3.73$ cu. ft. per lb.	$v_2 = 5.52$ cu. ft. per lb.
$H_1 = 1189.6$ B.t.u.	$H_2 = 1185.0$ B.t.u.

Note: Since the temperature at (2) is greater than the temperature of dry saturated steam at 80 lb. per sq. in. abs. pressure by 5° F., the steam is superheated 5° F.

Now, from Eq. (53),

$$JH_1 + \frac{V_1^2}{2g} + Z_1 + J_1Q_2 = JH_2 + \frac{V_2^2}{2g} + Z_2$$

The pipe is horizontal so

$$Z_1 = Z_2$$

and since no heat is assumed to enter or leave the pipe

$$_1Q_2 = 0$$

$$J(1189.6) + \frac{V_1^2}{2g} = J(1185.0) + \frac{V_2^2}{2g}$$

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = 778(1189.6 - 1185.0) \tag{1}$$

From the equation of continuity

$$W = w_1A_1V_1 = w_2A_2V_2$$

or

$$\frac{A_1V_1}{v_1} = \frac{A_2V_2}{v_2}$$

Therefore $$V_2 = \frac{v_2}{v_1}V_1 = \frac{5.52}{3.73}V_1 = 1.48\,V_1$$

Substituting in (1)

$$\frac{(1.48\,V_1)^2}{2g} - \frac{V_1^2}{2g} = 778 \times 4.6$$

$$\frac{1.2V_1^2}{2g} = 778 \times 4.6$$

$$V_1 = \sqrt{\frac{778 \times 4.6 \times 2g}{1.2}} = 439 \text{ ft. per sec.}$$

$$W = w_1A_1V_1 = \frac{A_1V_1}{v_1} = \frac{\pi \times 6^2 \times 439}{4 \times 144 \times 3.73} = 22.9 \text{ lb. per sec.} \qquad \textit{Ans.}$$

Illustrative Problem 2. Carbon dioxide gas is flowing isothermally through a horizontal pipeline which changes in diameter from 6 in. to 3 in. At a point in the pipeline where the diameter is 6 in. the pressure is 25.3 lb. per sq. in. gage and the temperature is 60° F. At a point farther along the pipeline where the diameter is 3 in., the pressure is 20.3 lb. per sq. in. gage. Given the gas constant for carbon dioxide $R = 35.1$, and neglecting friction find:

(*a*) The quantity of gas flowing.

(*b*) The amount of heat supplied to the pipeline between the two points from an outside source.

(*a*) Equation (54) is

$$\frac{V_1^2}{2g} + Z_1 = \frac{V_2^2}{2g} + Z_2 + \int_{p_1}^{p_2} v\,dp \tag{1}$$

$$Z_1 = Z_2$$

For isothermal flow

$$pv = p_1v_1$$

$$v = \frac{p_1v_1}{p}$$

Therefore

$$\int_{p_1}^{p_2} v\,dp = p_1v_1 \int_{p_1}^{p_2} \frac{dp}{p} = p_1v_1 \Big[\log_e p\Big]_{p_1}^{p_2}$$

$$= p_1v_1 (\log_e p_2 - \log_e p_1)$$

$$= -p_1v_1 \log_e \frac{p_1}{p_2}$$

Substituting this value of $\int_{p_1}^{p_2} v\,dp$ in (1)

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = p_1v_1 \log_e \frac{p_1}{p_2} \tag{2}$$

From $$p_1v_1 = p_2v_2$$

and $$W = \frac{A_1V_1}{v_1} = \frac{A_2V_2}{v_2}$$

$$V_2 = \frac{v_2}{v_1}\frac{A_1}{A_2} V_1 = \frac{p_1}{p_2}\frac{A_1}{A_2} V_1 = \frac{p_1}{p_2}\frac{D_1^2}{D_2^2} V_1$$

The pressures must be absolute pressures in pounds per square foot.

Therefore $$V_2 = \left(\frac{25.3 + 14.7}{20.3 + 14.7}\right)\frac{144}{144}\left(\frac{6}{3}\right)^2 V_1$$

$$= 4.57V_1$$

From $$pv = RT$$

$$v_1 = \frac{RT_1}{p_1} = \frac{35.1 \times (60 + 460)}{40 \times 144} = 3.17 \text{ cu. ft. per lb.}$$

Substituting in (2)

$$\frac{[4.57V_1]^2}{2g} - \frac{V_1^2}{2g} = 40 \times 144 \times 3.17 \log_e \frac{40}{35}$$

$$\frac{19.9V_1^2}{2g} \quad 40 \times 144 \times 3.17 \times 0.134$$

$$V_1 = \sqrt{\frac{2g \times 40 \times 144 \times 3.17 \times 0.134}{19.9}}$$

$$= 89.0 \text{ ft. per sec.}$$

$$W = \frac{A_1V_1}{v_1} = \frac{\pi \times 6^2 \times 89.0}{4 \times 144 \times 3.17} = 5.50 \text{ lb. per sec.}$$ *Ans.*

(*b*) A solution for this part of the problem is made by examining Eq. (53),

$$JH_1 + \frac{V_1^2}{2g} + Z_1 + J_1Q_2 = JH_2 + \frac{V_2^2}{2g} + Z_2$$

By definition $$H = I + {}^{pv}$$

In the above problem the flow was isothermal and since I depends only upon the temperature

$$I_1 = I_2$$

Also since
$$pv = RT$$

and
$$T_1 = T_2$$

$$p_1v_1 = p_2v_2$$

Therefore
$$H_1 = H_2$$

Equation (53) reduces to

$$\frac{V_1^2}{2g} + J_1Q_2 = \frac{V_2^2}{2g}$$

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = J_1Q$$

But from (2)

$$J_1Q_2 = p_1v_1 \log_e \frac{p_1}{\cdot}$$

$$40 \times 144 \times 3.17 \times 0.134 = 2450 \text{ ft. lb. per lb.}$$

$$_1Q_2 = \frac{2450}{778} = 3.14 \text{ B.t.u. per lb. flowing}$$ *Ans.*

PROBLEMS

78. Steam flows through a 4-in. pipeline. At point A the temperature is 327.8° F., the pressure is 100 lbs. per sq. in. abs., and the steam has a quality of 0.99. At point B, further along the pipe, the temperature is 281.0° F. and the pressure is 50 lbs. per sq. in. abs. From steam tables the following data are obtained:

H_A 1177.4 B.t.u. per lb., $v_A = 4.38$ cu. ft. per lb.,

H_B 1173.6 B.t.u. per lb., $v_B = 8.51$ cu. ft. per lb.

If the amount of heat dissipated into the surroundings between the two points is estimated to be 1 B.t.u. per lb., find the weight flow through the pipe.

Ans. $W = 4.47$ lb. per sec.

79. Evaluate $\int vdp$ between p_1 and p_2 if the relationship between p and v is given by the equation

$$pv^k = p_1v_1^k$$

80. Air flows through a tube which changes in diameter from 8 in. to 6 in. At a point where the diameter is 8 in., the pressure is 45 lb. per sq. in. abs. and the temperature is 80° F. At a point where the diameter is 6 in., the pressure is 40 lb. per sq. in. abs. (*a*) Assuming that the flow between the two points takes place adiabatically and that the flow is frictionless, find the weight flow through the tube. For air, $R = 53.3$ and $n = 1.4$. (*b*) What are the velocity and temperature at the second point?

47. Friction in Bernoulli Equation for Compressible Fluids. — As has already been stated, the flow of fluids cannot take place without friction. Consequently, the external work supplied to a given mass of fluid is only partially available in changing the potential and kinetic energies of the mass; the remainder is converted into heat energy by friction. If only the kinetic and potential energies of the fluid are considered useful, as is the case with flowing liquids, the heat generated by friction is lost energy. In dealing with gases, energy is lost just as it is for liquids, but this energy can be included in the Q-term, and no friction term, as such, is needed. Thus, if all three forms of energy, kinetic, potential, and heat, are considered, as is necessary in dealing with compressible fluids, the result of friction is merely to increase those changes which are affected by the addition of heat and to reduce those terms which represent changes in the other forms of energy.

In order to make the above statements clear, we rewrite the differential equation for compressible fluids Eq. (51).

$$JdQ = vdp + dz + \frac{VdV}{g} + JdI + pdv$$

Assuming that the same amount of heat, dQ_1, from an external source is available, the only effect of the heat due to friction is to increase this to dQ and to produce compensating changes in the terms on the right side of the equation. Heat added to a gas is used in increasing its intrinsic energy and in doing work. The heat of friction, however, increases the terms JdI and pdv over what they would be in frictionless flow. The increase in these terms is partly compensated for by the changes represented in the other terms on the right side of Eq. (51). Thus, if the changes indicated by the differentials in Eq. (51) are the ones actually existing when flow takes place, the equation is applicable to flow with friction. This again integrates into

$$p_1v_1 + \frac{V_1^2}{2g} + Z_1 + JI_1 + J_1Q_2 = p_2v_2 + \frac{V_2^2}{2g} + Z_2 + JI_2$$

which was Eq. (52).

Equation (52) contains no term representing friction. An equation which is very useful in the flow of perfect gases involving friction may be derived in the following manner.

Referring to Eq. (52), it was stated that when actual flow takes place the terms JdI and pdv are greater than they would be if the flow were frictionless, due to the heat generated. It was also stated that the remaining

terms on the right side of the equation are smaller than they would be if the flow were frictionless. Thus a portion of the mechanical energy (kinetic and potential) has been converted into an exactly equal amount of heat. If we represent the former by dh_f in foot pounds per pound and the latter by dQ_f in B.t.u. per pound, we have

$$dh_f - JdQ_f = 0 \tag{55}$$

Adding Eq. (55) to the right side of Eq. (51), we obtain

$$JdQ = vdp + \frac{VdV}{g} + dZ + dh_f + JdI + pdv - JdQ_f$$

or
$$J(dQ + dQ_f) = vdp + \frac{VdV}{g} + dZ + dh_f + JdI + pdv$$

Now $(dQ + dQ_f)$ is the amount of heat supplied to the gas and produces the change.

$$dI + \frac{pdv}{J}$$

Therefore
$$J(dQ + dQ_f) = JdI + pdv$$

and
$$vdp + \frac{VdV}{g} + dZ + dh_f = 0 \tag{55a}$$

Integrating between points (1) and (2) along the flow

$$\frac{V_1^2}{2g} + Z_1 = \frac{V_2^2}{2g} + Z_2 + \int_{p_1}^{p_2} vdp + {}_1h_{f_2} \tag{56}$$

Equation (56) is the Bernoulli equation for compressible fluids including friction. Its use depends upon the possibility of determining $\int_{p_1}^{p_2} vdp$ and ${}_1h_{f_2}$. If v can be expressed as a function of p, the first of these may be obtained. Methods for obtaining h_f in certain types of flow of compressible fluids are discussed in Art. 99, Chap. VIII. In the problems of this chapter, h_f will be given or will be the only unknown to be calculated.

Illustrative Problem. Air flows isothermally at the rate of 2 lb. per sec. through a horizontal pipeline which changes from 2 in. to 3 in. in diameter. At a point in the 2 in. portion, the pressure is 50 lb. per sq. in. abs., at another point in the 3 in. portion, the pressure is 52 lb. per sq. in. abs. R for air is 53.3. $T = 90°$ F. Find (*a*) the loss of mechanical energy due to friction. (*b*) the heat transfer.

(*a*) By use of Eq. (56)

$$\frac{V_1^2}{2g} + Z_1 = \frac{V_2^2}{2g} + Z_2 + \int_{p_1}^{p_2} v\,dp + {}_1h_{f_2}$$

From the equation of continuity,

$$W = w_1A_1V_1 = w_2A_2V_2$$

and

$$pv = RT$$

$$v_1 = \frac{53.3\,(90 + 460)}{50 \times 144} = 4.07 \text{ cu. ft. per lb.}$$

$$v_2 = \frac{53.3\,(90 + 460)}{52 \times 144} = 3.92 \text{ cu. ft. per lb.}$$

$$V_1 = \frac{W}{w_1A_1} = \frac{Wv_1}{A_1} = \frac{2 \times 4.07}{.0218} = 373 \text{ ft. per sec.}$$

$$V_2 = \frac{Wv_2}{A_2} = \frac{2 \times 3.92}{.0492} = 159 \text{ ft. per sec.}$$

Since the pipe is horizontal

$$Z_1 = Z_2$$

For isothermal conditions (see Illustrative Problem, p. 88)

$$\int_{p_1}^{p_2} v\,dp = p_1v_1 \log_e \frac{p_2}{p_1}$$

$$50 \times 144 \times 4.07 \log_e \frac{52}{50}$$

$$= 50 \times 144 \times 4.07 \times 0.0392$$

$$= 1{,}149 \text{ ft. lb.}$$

Substituting the above values in Eq. (56)

$$\frac{373^2}{2g} - 1{,}149 = \frac{159^2}{2g} + {}_1h_{f_2}.$$

$${}_1h_f = \frac{139{,}100 - 25{,}300}{2g} - 1{,}149$$

$${}_1h_{f_2} = 1{,}767 - 1{,}149 = 618 \text{ ft. lb. per lb.}$$

(*b*) From Eq. (52)

$$JH_1 + \frac{V_1^2}{2g} + Z_1 + J_1Q_2 = JH_2 + \frac{V_2^2}{2g} + Z_2$$

Since the flow is isothermal

$$H_1 = H_2$$

Substituting known values for V_1 and V_2

$$\frac{373^2}{2g} + J_1Q_2 = \frac{159^2}{2g}$$

$$_1Q_2 = \frac{159^2 - 373^2}{2g \times 778} = -2.27 \text{ B.t.u. per lb.} \qquad \textit{Ans.}$$

PROBLEMS

81. Oxygen flows isothermally through a 2-in. pipe at the rate of 1.2 lb. per sec. The temperature is 40° F. At point A, the pressure is 50 lb. per sq. in. abs. At point B, the pressure is 20 lb. per sq. in. abs. What is the friction loss in this length of pipe? For oxygen, $R = 48.2$ and $n = 1.40$.

CHAPTER VI

DIMENSIONAL ANALYSIS

48. Introduction. — It has been stated that many of the problems encountered in the field of fluid mechanics cannot be solved by purely theoretical analysis because of the complex nature of the phenomena involved. Since this is true, experimental investigation is indispensable for determining the laws governing the motion of fluids.

Quite often, the results of an experiment are expressed by an equation or curve which shows the relationship between certain measurable quantities and other quantities the values of which are desired. However, before an experimental project is begun, the investigator should decide what quantities have a bearing on the phenomena and the general form of the equation by which his results are to be expressed or, if a curve is to be used, what values should be plotted to give the best picture of the relationship between the several variables. When the variables involved are known, it is possible to predict the form of the mathematical equation. The process by which this is done is known as *dimensional analysis*.

The fundamental ideas of dimensions originated with Sir Isaac Newton and were published by him in his " Principia " some 250 years ago. However, it remained for Fourier, Stokes, Lord Rayleigh, Reynolds, and others to develop the method and make it applicable for analyzing engineering problems. The greatest single step towards making the method of use to engineers was taken by Buckingham.[1]

The purpose of the analysis of a given problem is to determine the manner in which the variables can best be related and thereby reduce the number of separate quantities which must be varied in the later experimental investigations. It is a method of transforming mathematical equations. However, one must know the number of variables appearing and should any of importance be omitted, the dimensional reasoning cannot supply it. In other words, one wholly unfamiliar with the problem cannot successfully study it by this method.

49. Definitions. — One speaks of the dimensions of a building and, in so doing, conveys an idea of the physical magnitude of the structure. The building measures so many units of length (feet) in the different directions.

[1] Buckingham, " Model Experiments and the Forms of Empirical Equations," *T.A.S.M.E.*, V. 37, pp. 263–296.

This meaning is the one which is most commonly associated with the word dimension; but there is another more far-reaching idea contained in the use of the word. We speak of an area and visualize a product of two lengths; or speak of a velocity and visualize a given distance, or length, passed over in a unit of time. Now the dimension of an area is a length squared, and of a velocity is a length divided by time. It is this concept of the word *dimension* which is now under consideration.

The numerical value of the physical quantities depends upon the magnitude of the unit of measure. Thus an area may be expressed as one square inch, one square foot, one acre or one square mile. In each case, there has been one unit of area, but the physical magnitude was very different due to the variation in the yard stick that was used.

A *dimensional* quantity is any quantity which can be expressed in terms of one or more of the fundamental dimensions which we might choose. The dimensions of length, time, and either mass or force are sufficient for the solution of the problems encountered in fluid mechanics. The dimension of mass will be used in this text rather than force. Thus, the dimensions to be used are:

1. Length
2. Time
3. Mass

A *dimensionless* quantity is one whose numerical value does not change when the size of the fundamental unit is changed so long as the relation between the derived and fundamental units is not changed. The units in the various terms must be consistent. A consistent set of units can be defined by the units of the fundamental dimensions in the Newton equation $F = Ma$. For example, if the engineering system of units is to be used, the force unit would be in pounds, the mass unit in slugs, the length unit in feet and the time unit in seconds. For any other system, the corresponding units of that system would be used. A dimensionless quantity may be a ratio, as the slope of a roof, the angle of a sector of a circle when measured in radians, and, as will be shown later in this chapter, certain ratios which are obtained by dimensional reasoning.

50. Units and Dimensions. — The measurement of the physical quantities with which we deal in this text can be made with the three fundamental dimensions which are independent. These dimensions are length, time, and mass, represented respectively by L, T, and M. The dimension of mass has been chosen since it is a more fundamental quantity than force. The English system of units, namely the foot-pound-second system, will be used where pound is a unit of force and not the unit of mass.

51. Derived Dimensions. — Derived dimensions are those which depend upon the choice of the fundamental dimensions. Certain derived dimensions will now be obtained in order to explain the process.

An *area* is the portion of a surface the magnitude of which is proportional to the product of two lengths both of which are measured by the same scale. Thus:

$$[\text{Area}] = L \times L = L^2$$

Where [Area] shall be read " the dimensions of an area."

Velocity is the time rate of linear displacement and is expressed by the equation

$$V = \frac{ds}{dt}$$

The fact that these quantities are differential does not alter the condition that a velocity is a distance divided by a time.

$$[\text{Velocity}] = \frac{L}{T} = LT^{-1}$$

Acceleration is the time rate of change of a velocity and is expressed by the equation

$$a = \frac{dv}{dt}$$

$$[\text{Acceleration}] = \frac{L}{T} = LT^{-2}$$

Force is described as that quantity which tends to produce a change in the velocity of a body. It is defined as the product of the mass times the acceleration.

$$[\text{Force}] = [\text{Mass} \times \text{Acceleration}]$$
$$= M \times LT^{-2} = MLT^{-2}$$

Viscosity is defined as that property of a liquid or gaseous body which causes it to offer more or less resistance to all efforts to produce a relative displacement of the portions of the body with reference to each other. This definition was expressed in equation form as

$$F = \mu A \frac{dv}{dy} \tag{26}$$

Solving (26) for μ, we obtain

$$\mu = \frac{Fdy}{Adv}$$

from which

$$[\text{Viscosity}] = \frac{[\text{Force} \times \text{Distance}]}{[\ \text{Area} \times \text{Velocity}\ ]}$$

$$\frac{MLT^{-2} \times L}{L^2 \times LT^{-1}} = ML^{-1}T^{-1}$$

The dimensions of other quantities may be obtained in a similar manner. Table III gives the units of a number of quantities which appear in fluid mechanics formulas.

TABLE III: DIMENSIONS OF PHYSICAL QUANTITIES

Quantity	*Symbol*	*Common English Unit*	*Dimensions*
Geometric:			
Length	L	Ft.	L
Area	A	Ft.^{2}	L^2
Volume	V	Ft.^{3}	L^3
Head	H	Ft.	L
Kinematic:			
Time	T	Sec.	T
Revolutions per minute	n	r.p.m.	T^{-1}
Velocity	V	Ft. per sec.	LT^{-1}
Angular velocity	ω	Radians per sec.	T^{-1}
Acceleration	a	Ft. per sec.^{2}	LT^{-2}
Gravity	g	Ft. per sec.^{2}	LT^{-2}
Angular acceleration	α	Radians per sec.^{2}	T^{-2}
Kinematic viscosity	ν	Ft.^{2} per sec.	L^2T^{-1}
Discharge	Q	Ft.^{3} per sec.	L^3T^{-1}
Dynamic:			
Mass	M	Slug	M
Force	F	Lb.	MLT^{-2}
Surface tension	σ	Lb. per ft.	MT^{-2}
Density	ρ	Slug per ft.^{3}	ML^{-3}
Specific weight	w	Lb. per ft.^{3}	$ML^{-2}T^{-2}$
Viscosity	μ	Lb. sec. per ft.^{2}	$ML^{-1}T^{-1}$
Pressure	p	Lb. per ft.^{2}	$ML^{-1}T^{-2}$
Modulus of elasticity	K	Lb. per ft.^{2}	$ML^{-1}T^{-2}$
Impulse (Momentum)	I (M)	Slug ft. per sec.	MLT^{-1}
Work (Energy)	W (E)	Ft. lb.	ML^2T^{-2}
Power	P	Ft. lb. per sec.	ML^2T^{-3}

PROBLEMS

82. Solve for the dimensions of kinematic viscosity.
83. Solve for the dimensions of specific weight.
84. Solve for the dimensions of work.
85. Solve for the dimensions of power.

52. Applications of Dimensional Reasoning. — In order to illustrate the Rayleigh method by which the form of engineering equations may be obtained by dimensional reasoning, certain examples will be solved. Let it be required to find the form of the equation for the period of a simple pendulum. We will assume that the period, t, depends upon the product of some powers of the length of the supporting cord, l, the mass of the bob, M, and the gravitational field, g, in which the motion occurs. We then obtain

$$t \propto M^x l^y g^z \tag{57}$$

Where $\propto$ is read "varies as." Substituting the units of the various quantities into the above expression, it follows that

$$[T] = [M^x][L^y][LT^{-2}]^z$$

or

$$T = M^x L^y L^z T^{-2z} \tag{58}$$

If Eq. (58) is homogeneous, the corresponding terms on the two sides of the equation must have like exponents. Equating the exponents for these terms, we have

$$\begin{aligned} &\text{For mass} && 0 = x \\ &\text{For time} && 1 = -2z \\ &\text{For length} && 0 = y + z \end{aligned}$$

Solving the above equations for the values of x, y and z, we find $x = 0$, $y = \frac{1}{2}$ and $z = -\frac{1}{2}$. Substituting the values of the exponents in Eq. (57), we obtain

$$t \propto \sqrt{\frac{l}{g}}$$

or

$$t = c\sqrt{\frac{l}{g}} \tag{59}$$

We now see that the equation of the pendulum could be determined completely by a variation of only the length. The required experimentation has been greatly reduced because of the analysis.

We will pass now to a consideration of the resistance offered to the flow of a fluid. It is reasonable to assume that the resistance per unit area for a given surface will depend upon the velocity of flow, the density and viscosity of the fluid and the size of the surface. The equation may now be written

$$\tau \propto V^a \rho^b \mu^c D^d \tag{60}$$

where τ is the resistance per unit area of the interior of a pipe, say, V is the mean velocity, ρ the density of the fluid, μ the viscosity and D the diameter

of the pipe. Substituting the dimensions of the various expressions, we obtain the dimensional equation

$$ML^{-1}T^{-2} = L^aT^{-a}M^bL^{-3b}M^cL^{-c}T^{-c}L^d$$

Equating the exponents of corresponding variables,

$$\begin{aligned} &\text{For mass} & 1 &= b + c \\ &\text{For time} & -2 &= -a - c \\ &\text{For length} & -1 &= a - 3b - c + d \end{aligned}$$

We have three equations containing four unknowns, so it is not possible to obtain numerical values for the four exponents. It is possible to solve for three of the exponents in terms of the fourth. The values of a, b, and d will be determined in terms of c.

$$a = 2 - c$$

$$b = 1 - c$$

$$d = -c$$

Substituting these values in Eq. (60),

$$\tau \propto V^{2-c}\rho^{1-c}\mu^cD^{-c}$$

This expression may be rewritten

$$\tau \propto \rho V^2\left(\frac{\mu}{VD\rho}\right)^c \tag{61}$$

The expression $(\mu/VD\rho)$ is normally inverted and written $(VD\rho/\mu)$. In this form it is known as Reynolds number for which the symbol **R** will be used. The dimensions of the quantities which appear in the numerator of Reynolds number are the same as those appearing in the denominator. **R** is therefore dimensionless and has the same numerical value regardless of whether or not the English or c.g.s. system of units has been used in the measurement of the quantities. Equation (61) could be written

$$\tau = c\rho V^2\phi(\mathbf{R}) \tag{62}$$

In words, it can be said that for a given surface, the resistance per unit of area is directly proportional to the density, the square of the velocity and some function of the Reynolds number. The attention of the reader is directed to the fact that the shearing force does not truly vary as the square of the velocity. This is evident from an investigation of Eq. (61). From Eq. (62), it appears that the force varies as the square of the velocity only because a portion of the exponent is concealed in the function of the Reynolds number.

Had some other solution of the exponents been made as, for example, the solution of b, c and d in terms of a, a correct form of equation would have been obtained but it would not have been similar to Eq. (62) and would not have given a clear picture of the law of variation. One must possess a knowledge of the desired form of equation.

PROBLEMS

86. Solve for the form of Eq. (60) by obtaining the values of b, c and d in terms of a.

87. Given that the velocity of a liquid flowing from an orifice depends upon gravity, the head on the orifice and upon the viscosity and density of the liquid. Show that

$$V = C\sqrt{gH}\,\phi\left(\frac{g^{1/2}H^{3/2}}{\nu}\right)$$

88. Given that the pressure drop, P, in a pipe varies as the density and viscosity of the fluid, the mean velocity of flow and the diameter and length of the pipe. Using the expression

$$P \propto L^a d^b V^c \mu^d \rho^e$$

Show that

$$P \propto \rho V^2 \left(\frac{L}{d}\right)^a (\mathrm{R})^{-d}$$

53. The Buckingham π-Theorem. — Buckingham, in his article,[1] states that in order to interpret the experiments on models, " we must have an equation which describes the behavior of the machine or structure under conditions of service, and which contains as variables, the size of the machine and the quantities such as speed, applied forces, viscosity of the surrounding medium, etc., which suffice to specify all the essential circumstances of operation. Such an equation is called a physical equation."

Should one proceed by direct experimentation, the results could be obtained only after a long and expensive series of tests had been performed, providing the problem was at all complicated. Buckingham points out five advantages of applying the " principle of dimensional homogeneity ": First, it directs our attention to the things that we need to measure and may point out some simplifying approximations. Second, it reduces the number of separate variables. Third, it gives information as to the possible form of the empirical equation, and warns us not to blindly trust equations beyond the range of the experiments upon which they are based if they are not of this form. Fourth, it sometimes enables us to put the equation in such a form that we can use either English or Metric units indiscriminately without wasting time for conversion. And fifth, it often enables us to dispense with a complete experimental investigation, and

[1] E. Buckingham, " Model Experiments and the Forms of Empirical Equations," *T.A.S.M.E.*, V. 37, pp. 263-96.

shows us how a very incomplete set of experiments may give us reliable information.

Buckingham has shown that any equation

$$F(Q_1, Q_2, \cdots Q_n) = 0 \tag{63}$$

which describes a relationship between n different kinds of quantities $Q_1, Q_2, \cdots Q_n$ is always reducible to the form

$$f(\pi_1, \pi_2, \cdots \pi_{n-k}) = 0 \tag{64}$$

in which each of the variables π represents a dimensionless product of $k + 1$ Q-terms, n is the number of kinds of quantities and k is the number of independent variables which are needed in specifying the dimensions of the n quantities. Since length, time and mass are commonly needed in problems dealing with fluids, generally $k = 3$. We choose k Q-terms and let these appear, with some unknown exponent, in each of the π-terms. These Q-terms with the unknown exponents are then multiplied in turn by the remaining Q-terms with a unity exponent to form the different π-terms. The procedure is made clear by illustration in the next article.

The application of Eq. (64) normally gives three fewer π-terms than the original number of variables. It is because of this fact that the second advantage of applying dimensional reasoning can be realized. The form of the function, f, must be determined by experiment. It must be emphasized that each of the quantities, Q, which appear in Eq. (63) must be of a different kind. For example, if a number of lengths are included, they will not be different kinds of quantities and will not increase the number of π-terms. They will merely appear as ratios of lengths. Equation (64) can be imagined solved for any π-term as a new function of the remaining π-terms. Equation (64) is known as the π-theorem, the deduction of which is given in the appendix of Buckingham's article to which previous reference was made.

54. Application of the π-Theorem. — The following applications of the π-theorem will demonstrate the great advantage of its use as compared to the Rayleigh method. The advantage is the more pronounced the greater the number of different kinds of quantities which appear in the original function.

Let us consider the case of a liquid flowing in a smooth pipe. When the liquid is flowing at a constant rate, the pressure gradient P may be expected to depend on the diameter D, the mean velocity V, and the density ρ and viscosity μ of the liquid. If no important quantity has been omitted, it can be said that

$$F(P, D, V, \rho, \mu) = 0 \tag{65}$$

We have the following dimensional equations for the quantities appearing

in the function:

$$\text{Pressure gradient} = \frac{\text{force}}{\text{area}} \div \text{length} \quad \text{or } ML^{-2}T^{-2}$$

$$\text{Diameter} = \text{length} \qquad \text{or } L$$

$$\text{Velocity} = \text{length} \div \text{time} \quad \text{or } LT^{-1}$$

$$\text{Density} = \text{mass} \div \text{volume} \quad \text{or } ML^{-3}$$

$$\text{Viscosity} = \frac{\text{force}}{\text{area}} \div \text{velocity gradient or } ML^{-1}T^{-1}$$

In these dimensional equations, we find three independent dimensions, so $k = 3$ which gives $5 - 3 = 2$ π-terms. Therefore

$$f(\pi_1, \pi_2) = 0 \tag{66}$$

The meaning of the π-terms will now be illustrated. Due to the fact that we are primarily interested in the pressure gradient P, we would prefer to have it appear in only one of the terms, and we will choose to have the viscosity appear only in the other π-term. We will therefore write

$$\pi_1 = D^x V^y \rho^z P \tag{67}$$

$$\pi_2 = D^a V^b \rho^c \mu \tag{68}$$

Since these π-terms are dimensionless, the sum of the exponents of each of the fundamental dimensions in the terms on the right of the expressions must add to zero. Writing the dimensions for the quantities which appear in π_1, we obtain

$$\begin{aligned} \pi_1 &= (L^x)(L^yT^{-y})(M^zL^{-3z})(ML^{-2}T^{-2}) \\ &= L^{x+y-3z-2}T^{-y-2}M^{z+1} \end{aligned} \tag{69}$$

Since π_1 was dimensionless

$$\begin{aligned} x + y - 3z - 2 &= 0 \\ -y - 2 &= 0 \\ z + 1 &= 0 \end{aligned}$$

from which $x = 1$, $y = -2$ and $z = -1$. Substituting these values in Eq. (67), we have

$$\pi_1 = \frac{DP}{\rho V^2} \tag{70}$$

In the same manner we have for the dimensions of π_2

$$\begin{aligned} \pi_2 &= (L^a)(L^bT^{-b})(M^cL^{-3c})(ML^{-1}T^{-1}) \\ &= L^{a+b-3c-1}T^{-b-1}M^{c+1} \end{aligned} \tag{71}$$

Since π_2 is also dimensionless

$$a + b - 3c - 1 = 0$$
$$-b - 1 = 0$$
$$c + 1 = 0$$

from which $a = -1$, $b = -1$, $c = -1$. Substituting these values in Eq. (68), we have

$$\pi_2 = \frac{}{DV\rho} \tag{72}$$

Substituting the values of π_1 and π_2 in Eq. (66), we obtain

$$f\left(\frac{DP}{\rho V^2}, \frac{\mu}{DV\rho}\right) = 0 \tag{73}$$

Equation (73) can be imagined to be solved for any π-term in terms of a new function of those remaining. We shall solve for π_1.

$$\frac{DP}{\rho V^2} = \phi\left(\frac{DV\rho}{\mu}\right)$$

$$\phi(\mathrm{R})$$

or

$$P = \frac{\rho V^2}{D}\phi(\mathrm{R}) \tag{74}$$

In words, Eq. (74) tells us that the drop in pressure in a smooth pipe is proportional to the density of the liquid and the square of the velocity, inversely proportional to the diameter of the pipe, and that it depends upon some function of the Reynolds number. It is the form of this function that must be determined experimentally.

An equation, but one having an entirely different appearance, would have been obtained with a different choice of quantities in the π-terms. To illustrate this, let us take

$$\pi_1 = D^x V^y \mu^z P \tag{75}$$
$$\pi_2 = D^a V^b \mu^c \rho \tag{76}$$

Proceeding in a similar manner to that already used

$$\pi_1 = (L^x)(L^y T^{-y})(M^z L^{-z} T^{-z})(ML^{-2}T^{-2})$$
$$= L^{x+y-z-2} T^{-y-z-2} M^{z+1}$$

Therefore

$$x + y - z - 2 = 0$$
$$-y - z - 2 = 0$$
$$z + 1 = 0$$

From which $x = 2$, $y = -1$, $z = -1$. Substituting these values in Eq. (75), we obtain

$$\pi_1 = \frac{D^2P}{V\mu} \tag{77}$$

Also
$$\pi_2 = (L^a)(L^bT^{-b})(M^cL^{-c}T^{-c})(ML^{-3})$$
$$= L^{a+b-c-3}T^{-b-c}M^{c+1}$$

From this, we obtain $a = 1$, $b = 1$, $c = -1$. Substituting these values in Eq. (76), we obtain

$$\pi_2 = \frac{DV\rho}{\mu} \tag{78}$$

Substituting the values of π_1 and π_2 in Eq. (66) and solving for the desired quantity P similar to the method of solution used in obtaining Eq. (74), we obtain

$$P = \frac{V\mu}{D^2}\phi(\mathrm{R}) \tag{79}$$

It is evident that Eq. (79) furnishes an entirely different picture from that given by Eq. (74), and impresses upon one that care must be exercised in the choice for the quantities which appear in the different π-terms.

A comparison of the Rayleigh method and the π-theorem will show that the Rayleigh method would tend to become very tedious and unwieldy as the number of physical quantities needed to describe the process becomes greater. The π-theorem does not possess this disadvantage. An increase in the number of quantities increases the number of π-terms needed, but this increase does not make the solution of a given π-term any more difficult.

A more general flow problem will now be considered. Let it be assumed that the flow is influenced by two linear dimensions a and b; a velocity V; a pressure P; and the density ρ, specific weight w, viscosity μ, and surface tension σ of the fluid. We then have

$$f(a, b, V, P, \rho, w, \mu, \sigma) = 0 \tag{80}$$

We have here eight quantities of which two are of the same type. Three fundamental dimensions are needed to describe these quantities, so there will be $7 - 3 = 4$ π-terms plus a ratio of the two length terms. For the sake of demonstrating that this is the case, it will be assumed that there will be five π-terms. Let us take a, V and ρ as the repeating quantities in obtain-

ing the π-terms. Then we have

$$\pi_1 = a^{x_1} V^{y_1} \rho^{z_1} b$$

$$\pi_2 = a^{x_2} V^{y_2} \rho^{z_2} P$$

$$\pi_3 = a^{x_3} V^{y_3} \rho^{z_3} w$$

$$\pi_4 = a^{x_4} V^{y_4} \rho^{z_4} \mu$$

$$\pi_5 = a^{x_5} V^{y_5} \rho^{z_5} \sigma$$

Substituting the dimensions in π_1, we obtain

$$\pi_1 = L^{x_1} L^{y_1} T^{-y_1} M^{z_1} L^{-3z_1} L$$

From which $x_1 = -1$, $y_1 = 0$ and $z_1 = 0$

and $$\pi_1 = \frac{b}{a}$$

as we had already predicted. In a similar manner

$$\pi_2 = L^{x_2} L^{y_2} T^{-y_2} M^{z_2} L^{-3z_2} M L^{-1} T^{-2}$$

From the above, we obtain $x_2 = 0$, $y_2 = -2$, and $z_2 = -1$. Using these values, we obtain

$$\pi_2 = \frac{P}{V^2 \rho}$$

$$\pi_3 = L^{x_3} L^{y_3} T^{-y_3} M^{z_3} L^{-3z_3} M L^{-2} T^{-2}$$

from which $x_3 = 1$, $y_3 = -2$, and $z_3 = -1$. We then have

$$\pi_3 = \frac{aw}{V^2 \rho}$$

$$\pi_4 = L^{x_4} L^{y_4} T^{-y_4} M^{z_4} L^{-3z_4} M L^{-1} T^{-1}$$

from which $x_4 = -1$, $y_4 = -1$, $z_4 = -1$. Therefore

$$\pi_4 = \frac{\mu}{aV\rho}$$

Similarly $$\pi_5 = L^{x_5} L^{y_5} T^{-y_5} M^{z_5} L^{-3z_5} M T^{-2}$$

The general equation for this problem can now be written

$$f\left(\frac{b}{a}, \frac{P}{V^2\rho}, \frac{aw}{V^2\rho}, \frac{\mu}{aV\rho}, \frac{\sigma}{aV^2\rho}\right) \tag{86}$$

In the expression for π_3, the unit weight of the liquid appeared. This quantity is equal to the density times the acceleration due to gravity, or

$$\rho g = w$$

With this substitution,

$$\pi_3 = \frac{ga}{V^2}$$

This dimensionless number gives a measure of the wave making resistance of an object and is commonly called Froude's number, **F**, in honor of Mr. Froude who studied the resistance of ships. π_4 is the Reynolds number and π_5 is Weber's number which will be symbolized by **W**. Actually these numbers are generally written as the inverse of the qualities which were obtained.

Imagining Eq. (86) solved for π_2 we have

$$P = \rho V^2\phi(\mathbf{R},\mathbf{F},\mathbf{W},r) \tag{87}$$

PROBLEMS

89. Given that the shearing force per unit area at the periphery of a pipe varies as the mean velocity of flow, the density and viscosity of the fluid, the diameter of the pipe and the size of the roughness particles, show that

$$\rho V^2\phi\left(\mathbf{R}, \frac{e}{D}\right)$$

where **R** is the Reynolds number and e/D is the roughness ratio.

90. The loss of head in a pipe due to friction is expressed by

$$h_f = \phi(\nu, V, g, D, L, e)$$

where h_f is in feet of the fluid, ν is the kinematic viscosity of the fluid, V the mean velocity, g the acceleration due to gravity, D the diameter, L the length of the pipe, and e the measure of the roughness. Obtain the dimensional equation.

CHAPTER VII

MEASURING DEVICES

55. Introduction. — In engineering practice, it is often necessary to measure the discharge, or quantity, of fluid flowing past a given point, or section, in a channel. A number of devices have been developed for this purpose. The discharge is obtained by volumetric measurement with certain of these; while in others, the discharge is a function of the head, or of the difference of two pressures. Those in the first class are known as *quantity meters*, while those in the second are known as *rate meters*. The first class depends upon a positive measurement of volume or weight; while in the second class, the flow through the device is continuous and the determination of the discharge is dependent upon a calibration or coefficient curve.

The discussion has been divided into two main parts: that concerning the measurement of non-compressible fluids, and that concerning compressible ones. In general, the liquids fall in the first class, while gases fall in the latter. Distinct equations are needed in the measurement of the flow of gases due to the fact that the density changes with changes of pressure or temperature. Differences from these causes are normally not of great importance when a liquid is under consideration, and for this reason the equations are less complicated.

Two types of discharge will be considered, namely: *free discharge* and *submerged discharge.*

Free discharge exists whenever the fluid flows through the device into a fluid of negligible density. Water discharging into air, or air discharging under considerable pressure from a pipe through a nozzle into the atmosphere are examples of free discharge.

Submerged discharge exists whenever the fluid flows through the device into a fluid whose density is not negligible. The flow of a liquid in a pipe line which contains a reduced section, such that the reduced section constitutes the measuring device, is an example of submerged flow.

A. NON-COMPRESSIBLE FLUIDS

56. The Pitot Tube. — A simple instrument for measuring the velocity of a fluid is the Pitot tube, named after the inventor, Henri Pitot, a French engineer. The instrument was described by him in a paper[1] which was

[1] Pitot, H., "Description of a Machine for Measuring the Velocity of Flowing Water and the Speed of Vessels," *Mem. Acad. of Sci.* Paris, 1732.

published in 1732. A simple form of the instrument consists of a tube bent so that the end points directly against the flowing fluid, as shown in

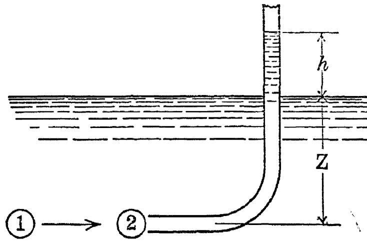


FIG. 75. Simple Pitot tube.

Fig. 75. Applying Bernoulli's equation between points (1) and (2), we obtain

$$\frac{V^2}{2g} + \frac{p_1}{w} + Z_1 = 0 + \frac{p_2}{w} + Z_1$$

Solving

$$\frac{V^2}{2g} = \frac{p_2}{w} - \frac{p_1}{w} = h$$

or

$$V = \sqrt{2gh} \tag{88}$$

From Eq. (88), it is evident that the pressure immediately before the opening of the tube, known as the point of stagnation, is greater than that in the body of the fluid by an amount which is equal to

$$\frac{wV^2}{2g} \qquad \rho\frac{V^2}{2}$$

The preceding expression could have been written

$$\Delta p = p_2 - p_1 = \frac{\rho V^2}{2} \tag{89}$$

It is evident when only one tube is used that it would be quite difficult to accurately read the height, h, that the fluid rises in the tube. To obviate this difficulty, a second tube is added as shown in Fig. 76. The opening, or openings, of the second tube are normal to the direction of the velocity. The tubes are connected to a manometer and the fluid drawn up into it by means of a suction. Should it be desired to measure the velocity of a gas, some gage fluid would be used in the manometer.

While Eq. (88) indicates that no coefficient is needed for a Pitot tube, such may not be the case. The reading will change with the form of the nose and the position of the static openings with reference to the nose and stem. For this reason, it is necessary that a tube be calibrated by holding

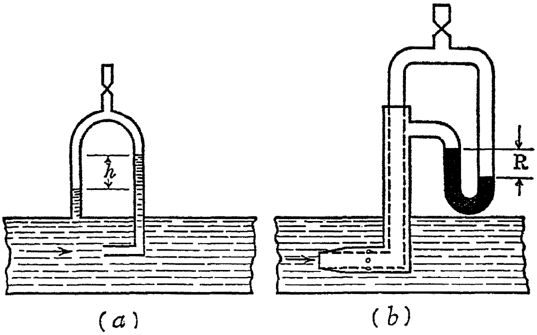


FIG. 76

it in a fluid moving with a known velocity comparable to that which is to be measured. By obtaining a number of such readings, a calibration curve for the tube may be prepared. The value of the actual velocity is obtained from the equation

$$V = c\sqrt{2gh} \tag{90}$$

with the value of c varying from about 0.8 up to possibly slightly in excess of unity. A value of 0.8 could be obtained on a small tube having a short nose with an overall length of one or two diameters. This short length would cause a tendency for a suction on the static openings with a corresponding increase in the observed head readings.

Based on the results of a rather comprehensive series of tests, Ower and Johansen[1], of England, conclude, " The static holes should be at least six diameters back from the base of the head, and the stem at least fifteen diameters down wind from the holes. In these circumstances the calibration of the instrument will be within wide limits independent of the shape of the head used, and will consequently be insensitive to relatively large variations in the form of the nose. Considerable latitude in the manufacture of duplicates will therefore be permissible without entailing the necessity for these to be calibrated. Further, an error not exceeding 0.25 per cent on velocity will be incurred if an instrument of this type is used in which the stem is absent.

[1] E. Ower and F. C. Johansen, "The Design of Pitot-Static Tubes," *Aeronautical Research Committee, Reports and Memoranda,* No. 981, London, 1925.

"From both aerodynamic and mechanical considerations a hemispherical nose appeared to be the most suitable, and this was incorporated on the modified form of instrument developed as a result of these experiments. The tests on this instrument gave satisfactory results; its calibration was found to be 0.3 per cent on velocity higher than that of the standard (one

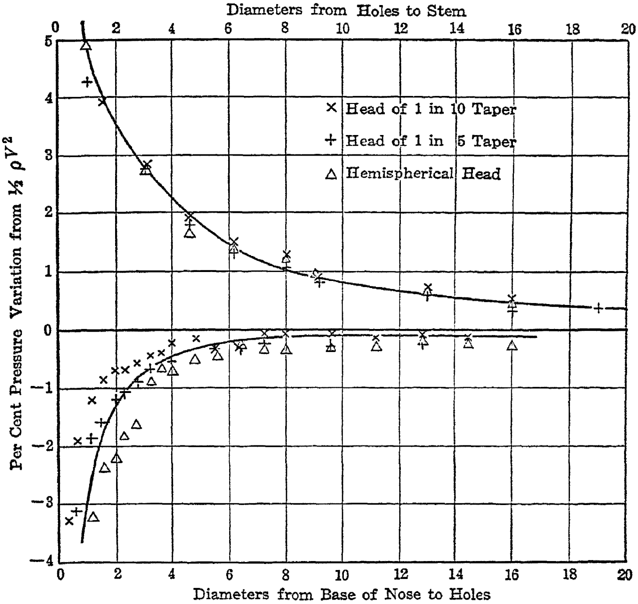


FIG. 77. Pressure changes due to the tube.

having a head tapered 1–10), and it was markedly less sensitive than the latter to angular deviations of the wind direction." Figure 77 shows the effect of the position of the static holes upon the static pressure reading. It can be seen that the nose causes a lowering of the pressure along the length of the tube while the stem causes an increase in pressure along the tube.

L. Prandtl[1] proposed a Pitot tube of such proportions that the reduced pressure caused by the flow past the nose would be exactly balanced by an increased pressure due to the presence of the stem. As a result, the Prandtl tube, the proportions of which are shown in Fig. 78, has a unity coefficient.

Both the English and the Prandtl tubes possess the disadvantage of being rather unwieldy for use where the streamlines are curved. Should it

[1] Prandtl and Tietjens, "Applied Hydro- and Aero-Mechanics," p. 230, McGraw-Hill Book Co., 1934.

be desirable to obtain the velocity at a point where the streamlines are curved, it would probably be preferable to use a more compact tube.

When a tube is to be used to measure the velocity of a turbulent fluid, the observed head reading will be somewhat high due to the fact that the rise is caused by the average of the squares of the instantaneous velocities that exist at the point in question. The average square of a number of quantities of varying magnitude is always greater than the square of the mean quantity. It is because of this effect and the effect of the shape of the tube upon the observed pressure difference that it is essential that the tube be calibrated.

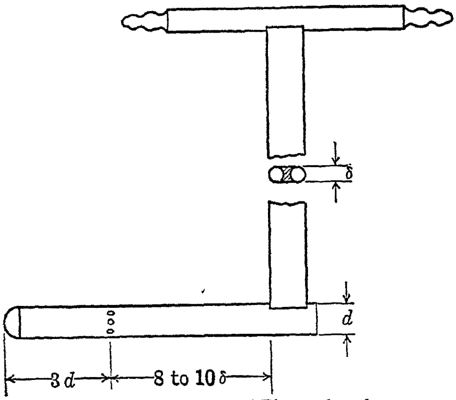


FIG. 78. The Prandtl Pitostatic tube.

While the Pitot tube can be used for the measurement of velocities in open channel work or in the open air for aviation, it is especially well adapted for use in measuring the velocities in a pipe operating under pressure. A hole is drilled and tapped in the pipe and the tube is inserted through a stuffing box. Readings are taken for a series of points on a diameter and the velocities are computed for these points. The mean velocity is then obtained by means of the n-point method where n would normally be taken equal to 10.

The 10-point method will now be described. A circle whose diameter is equal to that of the pipe is drawn and the area of this circle is divided into five equal parts by drawing the solid circles shown in Fig. 79*a*. Each of the five areas is then divided into two equal parts by means of the five broken circles. One-tenth of the original area lies within the smallest broken circle, and within each of the annular rings. Ten readings are taken from the velocity traverse at distances from the centerline of the pipe equal to the radii of the five broken circles, as shown in Fig. 79*b*. The arithmetic mean of these ten velocities is the desired mean velocity of flow in the pipe. The $n/2$ radii for the dotted circles could have been obtained mathematically as follows:

$$\pi r_i^2 = \left(\frac{1}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}\right) A \tag{91}$$

where r_i is the radius of any broken circle, n is the desired number of points

and A is cross-sectional area of the pipe. The mean of velocities from equally spaced points would give undue weight to the high velocities at the center of the pipe and too little weight to those near the wall of the pipe.

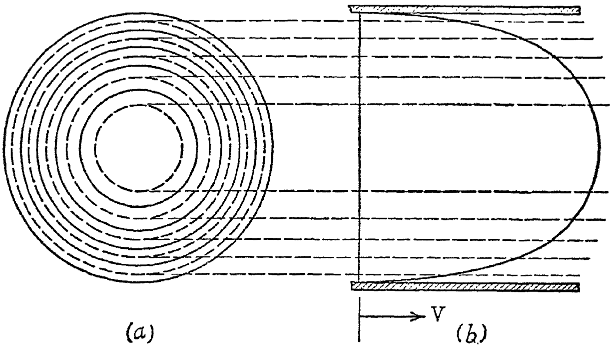


Fig. 79

PROBLEMS

91. A Pitot tube is to be calibrated in a jet of water having a velocity of 10.5 ft. per sec. Find the value of c if the manometer difference is 1.78 ft.

92. A Prandtl tube is used to measure the velocity of an air stream. The reading of an air-water manometer is 8 in. p_a = 12 lb. per sq. in. abs., T = 10° F. Find the velocity of the air stream. *Ans.* V = 197 ft. per sec.

93. The velocity traverse in a 6 in. pipe carrying water is symmetrical with respect to the geometric axis. The manometer readings at $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, $1\frac{3}{4}$, $2\frac{1}{4}$ and $2\frac{3}{4}$ inches from the axis are respectively 1.00, 0.97, 0.92, 0.78, 0.59, and 0.37 ft. of water. The coefficient of the tube is 0.96. Plot the velocity traverse and find the mean velocity by means of the 10-point method.

57. Orifices. — An orifice may be defined as a geometric opening in a thin wall, smaller than the conducting channel, through which a fluid discharges. The opening might be in the side, or bottom, of a tank; or it might be in a plate which would be placed either at the end of a pipe, or between two flanges in the pipe line. Orifices are classified according to the shape of the opening; for example, circular, rectangular, triangular, etc.; and according to the shape of the upstream edge of the opening; for example, sharp, bell-mouthed, etc. The circular orifice is preferable to the other shapes due to the ease of manufacture.

58. Free Discharge through Sharp-edged Orifices. — The sharp-edged orifice is one for which the upstream edge of the opening is a knife edge, or makes a 90° angle. The downstream edge may be beveled. These conditions are shown in Fig. 80. Orifices having a sharp edge are often

spoken of as "standard." The fluid will spring free at the edge and the discharge will be unaffected by the thickness of the plate so long as the jet does not again come in contact with the opening. The 90° edge has advantages over the beveled edge in that it is less subject to damage and it can be machined more accurately.

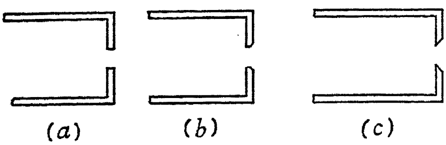


FIG. 80. Types of standard orifices.

Consider the orifice in the side of the tank of Fig. 81. The liquid in passing out through the orifice contracts for a distance of about one-half the orifice diameter. At this point, the streamlines of the jet are essentially parallel. For any cross section between the reduced section and the edge of the orifice, there is a pressure, greater than atmospheric, within the jet. This pressure is required in order to make the streamlines coming from positions parallel to the orifice plate turn and proceed normal to the plate. At and beyond the reduced section, there is a slight internal pressure which is caused by the surface tension around the jet. However, the pressure within the jet is normally considered atmospheric for the reduced section and for all points beyond it. Should the pressure within the jet at and beyond the reduced section be greater than that caused by the surface tension, an outward acceleration would be given the fluid particles and the area of the jet would become greater. Should the pressure be less than that caused by the surface tension, the area of the jet would continue to become smaller. The reduced section, B of Fig. 81, is known as the *vena contracta.*

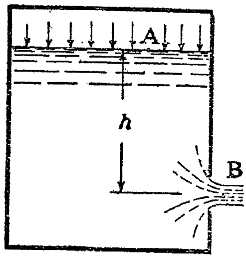


FIG. 81

Neglecting friction, the Bernoulli equation will now be written between points A and B.

$$\frac{V_A^2}{2g} + \frac{p_A}{w} + h = \frac{V_B^2}{2g} + \frac{p_B}{w}$$

Solving for V_B, we obtain

$$V_B = \sqrt{2g}\sqrt{h + \frac{p_A - p_B}{w} + \frac{V_A^2}{2g}} \tag{92}$$

If the tank is large in comparison to the size of the orifice and is open to the atmosphere, all except the first term under the second radical will disappear, and we have

$$V_B = \sqrt{2gh} \tag{93}$$

The fact that the velocity in the jet is the same as though the water particles had fallen freely through a height, h, was first demonstrated by Torricelli in 1643.

The value of V_B, which was obtained by Eq. (93), does not include the effect of any losses. There are small losses, and the actual value of V_B is obtained by correcting with a coefficient of velocity. Thus, we have

$$V_B = C_v\sqrt{2gh} \tag{94}$$

in which the value of C_v is usually between 0.98 and 1.00.

The value of the loss due to friction can be obtained by the substitution of h from Eq. (94) in the Bernoulli equation written between points A and B of Fig. 81, namely

$$h = \frac{V_B^2}{2g} + h_f$$

This last expression is obtained by considering friction in Eq. (93). Making the above substitution, we have

$$\frac{1}{C_v^2}\frac{V_B^2}{2g} = \frac{V_B^2}{2g} + h_f$$

or

$$h_f = \left(\frac{1}{C_v^2} - 1\right)\frac{V_B^2}{2g} \tag{95}$$

The expression for the loss of head given by Eq. (95) is very important as it will reappear in the later discussion of the flow through tubes, nozzles, and pipes.

The area of the jet at the vena contracta is expressed by the product of a coefficient of contraction and the area of the orifice opening. Thus,

$$A_B = C_cA$$

where A is the cross-sectional area of the orifice opening. The value of C_c varies between wider limits than C_v. For a very small orifice, it is approximately unity; but, as the size of the orifice becomes greater, it quickly drops and reaches a minimum value of about 0.61. The value normally ranges between 0.65 and 0.61 for the usual heads and diameters encountered. The discharge through the orifice is the product of A_B and V_B.

$$Q = A_BV_B = (C_cA)(C_v\sqrt{2gh})$$

$$= C_dA\sqrt{2gh} \tag{96}$$

From Eq. (96), it can be seen that the coefficient of discharge, C_d, is given

by the expression

$$C_d = C_c C_v$$

The values of the coefficients which were mentioned above can be obtained experimentally by several different methods. Some of these methods will be outlined briefly.

The actual velocity at the vena contracta could be measured by means of a calibrated Pitot tube. The comparison of this velocity to that with no losses would yield the coefficient of velocity.

The velocity at the vena contracta could be computed from the shape of the trajectory of the jet issuing from an orifice cut in a vertical plate. The trajectory of the center line of the jet, neglecting air resistance, would define a second degree parabola for which the vena contracta would represent the position of the vertex. The position of one point in addition to that of the vena contracta would be observed. The time required for the fluid particles in the jet to move from the vena contracta to this point would be found by considering the time required for a body to fall from rest through the vertical distance from the vena contracta to the point. The velocity at the vena contracta would then be obtained by dividing the horizontal distance between the two points by the time thus obtained.

The velocity at the vena contracta could be obtained by applying the principle of impulse and momentum. For this experiment, the jet would issue from an orifice which was cut in the vertical side of a tank. For a condition of no flow, the tank would be balanced and supported from a fulcrum. An unbalanced force would exist after flow began due to the fact that it would be necessary to create the momentum of the moving liquid, but the tank would be kept in balance by the addition of a weight on a known lever arm. In the general equation, $Ft = MV$, M is equal to ρQt. Replacing M by its equivalent, we have

$$F = \rho QV \tag{97}$$

The force given by Eq. (97) is the unbalanced force mentioned above, and it is the moment of this force with respect to the fulcrum that was balanced by the addition of the weight on the lever arm. The force is found by observing the magnitude and position of the balancing weight, and the discharge is measured. Knowing these values, the velocity at the vena contracta may then be obtained by the application of Eq. (97).

Once the velocity and measured discharge are known, the coefficient of contraction can be found by computation. This coefficient could also be obtained directly by carefully calipering the jet at the vena contracta and then comparing the area of the jet to that of the orifice opening. An accurate determination of the coefficient by direct measurement would be difficult for jets of small size.

PROBLEMS

94. The head on a 3 in. circular orifice is 20 ft. The velocity at the vena contracta is measured by means of a Pitot tube whose coefficient is 0.97 and the observed differential on a water-mercury manometer is 1.6 ft. Find C_d, C_v, and C_c if the measured discharge is 1.08 c.f.s.

Ans. $C_d = 0.613$; $C_v = 0.972$; $C_c = 0.631$.

95. The diameter of the jet issuing from a 2 in. orifice was found to be 1.65 in. The measured discharge under a head of 2 ft. was 9.8 cu. ft. in a one-minute period. Find C_d, C_v and C_c for this condition.

96. The trajectory of a jet which issued from an orifice in a vertical plate dropped 4 ft. from the vena contracta in moving 12 ft. horizontally. The head on the orifice was 9.4 ft. Find the coefficient of velocity.

97. The jet issued from the 5 in. orifice of Fig. 82 which was 3 ft. below the supporting fulcrum. A force of 61 lb. on a 2 ft. lever arm was needed to balance the force of the jet of water. The measured discharge and head were 1.33 c.f.s. and 4 ft. respectively. Find C_d, C_v and C_c for this head.

Ans. $C_d = 0.61$; $C_v = 0.982$; $C_c = 0.621$.

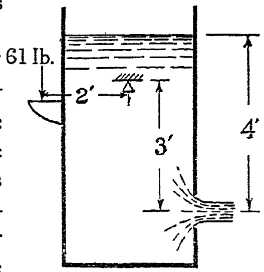


FIG. 82

59. Coefficients of Standard Orifices. — Many careful experiments have been performed on the flow through orifices. The results which have been obtained by the different experimenters vary somewhat. The divergence is more pronounced at low heads with small orifices for which it amounts to a few per cent. This range is partly due to a lack of similarity of the experimental apparatus, and partly to the difficulty of making accurate measurements of the small quantities involved. The variation in the published results does not detract from their value, but serves only to emphasize the need for a direct calibration of all apparatus of which great accuracy is expected.

The results of the tests show[1] that complete contraction cannot be obtained on small orifices. Smith and Walker found that for a given head the coefficient of contraction decreased with an increase in diameter up to a diameter of $2\frac{1}{2}$ inches after which further decrease was not noted. This same conclusion was reached by Bilton[2] in a paper read before the Victorian Institute of Engineers in 1908. Bilton concludes:

"1. The assumption that a coefficient of discharge common to all orifices from $\frac{1}{2}$ inch to 12 inches in diameter is reached at a head of 100 feet is erroneous.

[1] This fact is demonstrated by such data as were offered by Smith and Walker, *Proc. Inst. Mech. Engrs.*, p. 23, Jan. 1923.

[2] Bilton, H. J. I., "Coefficients of Discharge through Circular Orifices." Reviewed in *Eng. News*, July 9, 1908.

"2. In order to obtain complete and perfect contraction a certain minimum diameter and head are required. These appear to be approximately $2\frac{1}{2}$ and 17 inches, respectively.

"3. Orifices of $2\frac{1}{2}$ inches diameter and over, under heads of 17 inches and over, have a common coefficient of discharge, lying between 0.59 and 0.60 but probably about 0.598 (subject to the head's being not less than 2 or 3 diameters).

"4. In the case of orifices smaller than $2\frac{1}{2}$ inches in diameter, contraction is never perfect and complete under any head but is suppressed more and more as the diameter decreases, each size of orifice having its own constant or 'normal' coefficient of discharge and its own critical head.

"5. As the diameter decreases, the normal coefficient increases, as also the critical head.

"6. In an infinitely small orifice, contraction is entirely suppressed and unity becomes the coefficient of discharge for all heads (subject to the effect of capillarity, cohesion, viscosity, temperature, etc.).

"7. The discharge of a circular orifice under any given head is the same, whether the jet be horizontal, vertical, or at any intermediate angle."

The coefficients found by Smith and Walker for the 0.75 and 1.00 inch diameter orifices appear unduly large when compared with coefficients obtained by other experimenters. For their three larger sizes; namely 1.5, 2.0, and 2.5 inch diameter; the coefficients are more nearly in agreement with those found by others but average somewhat higher. The Smith and Walker values were obtained from average curves with individual points plotting as much as ± 2 per cent from the curve.

A table of coefficients of discharge for sharp-edged circular orifices was prepared by Smith[1] from the experiments of Poncelet and Lebros, T. G. Ellis, Hamilton Smith, Weisbach, Unwin, Francis, Steckel, Darcy, and Bazin. Selected values from Smith's table appear in Table IV.

Coefficients of velocity for water flowing from circular orifices do not vary greatly from unity and show a slight increase for greater heads. Table V gives the values of the coefficients of velocity as found by Smith and Walker.[2] From the small variation of these values, it is evident that a value of this coefficient can be predicted with a fair degree of accuracy for any given diameter of orifice.

Square orifices have been little used due to the difficulty of manufacturing, but their coefficients of discharge for the sharp-edged orifice do not seem to differ by more than one per cent from those for a circular orifice having the same diameter. Variations of this magnitude are not so great as the variations between individual tests by one experimenter, or between

[1] Smith, Hamilton, *Hydraulics*, p. 59, 1886.
[2] Smith and Walker, *op. cit.*

the mean results from different experimenters. Due to the difficulty of machining a square opening and due to the small number of tests which have been run for determining coefficients, it is therefore recommended that only circular orifices be employed.

TABLE IV. COEFFICIENTS OF DISCHARGE FOR CIRCULAR ORIFICES WITH FULL CONTRACTIONS

Head Over Center	*Diameter of Orifice, in Feet*							
Feet	0.02	0.04	0.05	0.10	0.15	0.20	0.40	1.00
0.5		0.633	0.627	0.615	0.605	0.600	0.596	
0.6	0.655	.630	.624	.613	.605	.601	.596	
0.8	.648	.626	.620	.610	.603	.601	.597	0.591
1.0	.644	.623	.617	.608	.603	.600	.598	.591
1.2	.641	.620	.615	.606	.602	.600	.598	.592
1.6	.636	.617	.612	.605	.601	.600	.599	.594
2.0	.632	.614	.610	.604	.600	.599	.599	.595
2.5	.629	.612	.608	.603	.600	.599	.599	.596
3.0	.627	.611	.606	.603	.600	.599	.599	.597
4.0	.623	.609	.605	.602	.599	.599	.598	.596
5.0	.621	.608	.605	.601	.599	.598	.598	.596
7.0	.616	.606	. 603	.600	.599	.598	.598	.596
10.0	.611	.603	.601	.598	.597	.597	.597	.595
20.0	.601	.599	.598	.596	.596	.596	.596	.594
50.0?	.596	.595	.595	.594	.594	.594	.594	.593
100.0?	.593	.592	.592	.592	.592	.592	.592	.592

TABLE V. COEFFICIENT OF VELOCITY C_v FOR CIRCULAR SHARP-EDGED ORIFICE

Head Over Center	*Diameter of Orifice, in Inches*				
Feet, H	0.75	1.0	1.5	2.0	2.5
0.2	0.951	0.955	0.958	0.967	0.980
0.4	.951	.956	.964	.973	.983
0.8	.953	.960	.971	.977	.986
1.2	.955	.963	.975	.983	.989
1.6	.956	.965	.978	.984	.989
2.0	.957	.966	.980	.984	.990
4.0	.956	.973	.983	.984	.990
8.0	.951	.977	.985	.984	.990
10.0	.953	.977	.985	.986	.990
20.0	.953	.978	.988	.986	.993
40.0	.954	.978	.990	.988	.993
60.0	.954	.979	.990	.988	.993
80.0	.954	.979	.992	.990	.993
100.0	.954	.979	.992	.990	.993

PROBLEMS

98. A sharp-edged orifice in the side of a large tank is 2 in. in diameter and discharges water under a head of 25 ft. Determine the velocity of the jet at the vena contracta and the rate of discharge.

99. Find the mean pressure within the jet in the plane of the orifice for the conditions stated in Prob. 98.

100. Find the diameter of a standard orifice which will discharge 0.8 c.f.s. under a head of 18 ft. *Ans.* $d = 2.69$ in.

101. Find the head required for a standard 3 in. orifice to discharge 2 c.f.s.

102. A circular orifice, for which $C_c = 0.63$ and $C_v = 0.98$, discharges vertically under a head of 16 ft. Find the height above the plane of the orifice to the point where the diameter of the jet is the same as the diameter of the orifice.

103. Find the head lost in the flow from the orifice of Prob. 102.

60. Dimensional Analysis of Free Orifice Flow. — Let us assume that the discharge Q from an orifice is dependent upon the velocity V at the vena contracta, some length d which would be the diameter of the orifice, the acceleration of gravity g and upon the viscosity μ, the density ρ and the surface tension σ of the liquid. This assumption can be written as the functional equation

$$f(V, d, \rho, Q, g, \mu, \sigma) = 0 \tag{98}$$

By the method discussed in Chap. V, we place

$$\pi_1 = V^{x_1} d^{y_1} \rho^{z_1} Q$$

$$\pi_2 = V^{x_2} d^{y_2} \rho^{z_2} g$$

$$\pi_3 = V^{x_3} d^{y_3} \rho^{z_3} \mu$$

$$\pi_4 = V^{x_4} d^{y_4} \rho^{z_4} \sigma$$

Solving for the different exponents, we find $\pi_1 = Q/Vd^2$; $\pi_2 = gd/V^2$; $\pi_3 = \mu/Vd\rho$; and $\pi_4 = \sigma/V^2 d\rho$.

The inverse of π_2, π_3 and π_4 is equal to Froude's, Reynolds' and Weber's numbers respectively and they refer to the wave making resistance, the effect of viscous forces and the effect of the force of surface tension upon the flow.

Since V is not an observed quantity, it is desirable to replace it with either $\sqrt{2gH}$ or $\sqrt{gH}$, which are dimensionally equivalent, in each of the π-terms. The d^2 in π_1 is proportional to the cross-sectional area, A, of the orifice and will be replaced by A. Making these substitutions, we have $\pi_1 = Q/A\sqrt{2gH}$, $\pi_2 = d/H$, $\pi_3 = \mu/\sqrt{2gH}d\rho$ and $\pi_4 = \sigma/gHd\rho$.

Equating π_1 to a function of the other three π-terms and solving for Q, we have

$$Q = A\sqrt{2gH}\,\phi(\mathbf{F}, \mathbf{R}, \mathbf{W}) \tag{99}$$

In order to determine the effect of any one of these numbers upon the magnitude of the coefficient of discharge, it would be necessary to conduct tests in such a way that the values of the other two would not change; or to show that for certain ranges, the effect of the other two was negligible. Should the coefficient of discharge be primarily a function of one of the dimensionless ratios and depend to a much smaller extent upon the values of the others, the coefficients of discharge plotted against this one ratio would produce a well defined curve. A good curve could not be obtained by use of the other ratios.

Let us consider the three dimensionless numbers which appeared in Eq. 99. Froude's number, $\mathbf{F} = H/d$, was obtained from the expression V^2/gd and varies as the square of the velocity and inversely as the diameter.

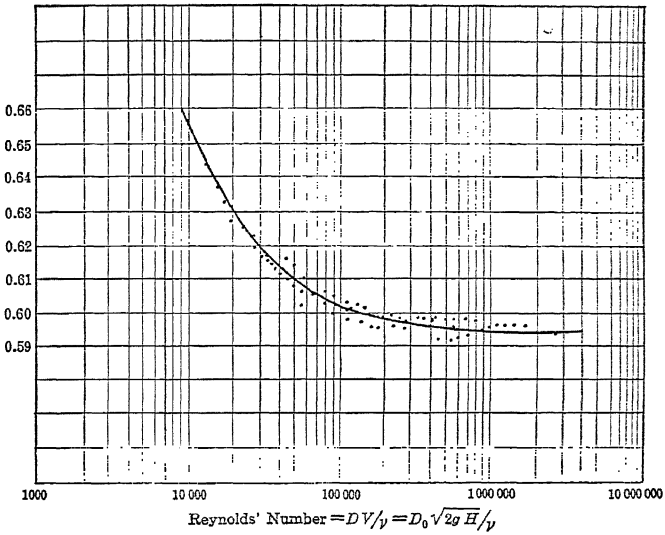


FIG. 83. Variation of the orifice coefficient of discharge with changes of the Reynolds number.

Referring to Table IV and realizing that the velocity in the jet is essentially constant for any one head, we see that for a constant velocity, C_d decreases as the diameter increases; also, for a constant diameter, C_d decreases as the velocity increases. Should it be possible to draw a well defined curve with $\mathbf{F}$ as one of the variables, C_d could not vary in this contradictory manner, so we conclude that the discharge does not depend mainly upon Froude's number.

The correct type of variation is shown by both Reynolds' and Weber's numbers, but the Reynolds criterion is better suited for this purpose due to the fact that surface tension assumes importance only for the low heads. Figure 83 shows the relationship between the coefficient of discharge and

the Reynolds number for all the values listed in Table IV with the exception of those for the 50 and 100 ft. heads. Smith considered the accuracy of these values doubtful, and they will not plot on the curve. The viscosity and density of the water may be somewhat in error due to the fact that a mean temperature of 60° F. was assumed in the computation of the Reynolds numbers.

The free orifice would not, in general, be used for liquids other than water. An orifice of this type could be used in the measurement of air flow during a fan test, say, but a discussion of the measurement of gas flow is considered in a later article of this chapter.

The coefficient of discharge can be found by the use of Fig. 83 for any fluid flowing through a free orifice. However, it will again be stated that a calibration would be needed if a high degree of accuracy were expected. The use of Fig. 83 should not introduce errors in excess of 2 per cent.

PROBLEMS

104. Find the coefficient of discharge for gasoline, $\nu = 0.000008$ ft.^{2} per sec., when flowing from a 1 in. orifice under a 20 ft. head. *Ans.* $C_d = 0.596$.

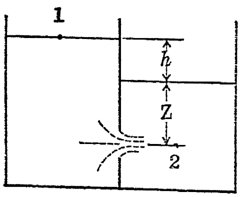


FIG. 84

105. Water at 40° F. flows through an orifice, $\frac{1}{2}$ in. in diameter, under a head of 15 ft. Find C_d.

106. Solve Prob. 105 for a temperature of 140° F.

61. Discharge of Submerged Orifices and Gates. — A submerged orifice is one for which the discharging medium flows into a channel containing the same medium the surface elevation of which is sufficient to affect the discharge. Only the condition of the jet being fully submerged will be considered here. A submerged orifice is shown in Fig. 84. The pressure at the vena contracta is assumed to be equal to that caused by the depth Z. Writing Bernoulli's equation between points (1) and (2) with friction neglected, we have

$$h + Z + \frac{V_1^2}{2g} = Z + \frac{V_2^2}{2g}$$

Solving for V_2,

$$V_2 = \sqrt{2g\left(h + \frac{V_1^2}{2g}\right)}$$

The area at the initial point is quite often large in comparison to the area of the orifice opening. Whenever this is the case, the last term under the radical is negligible and we have after the introduction of the coefficient C_v

$$V_2 = C_v\sqrt{2gh}$$

The discharge is obtained by the use of the orifice formula

$$Q = C_d A \sqrt{2gh} \tag{100}$$

The coefficients of discharge for the sharp-edged submerged orifices are approximately equal to those of similar orifices having free discharge. The student will use the Hamilton Smith coefficients in the solution of the problems on submerged orifices.

A gate differs from an orifice mainly in that the contractions are eliminated, or reduced, on a portion of the periphery of the gate opening. The opening of the gate can be varied and is generally rectangular or circular in shape. Due to the fact that the contractions are, in general, reduced, the coefficients of discharge depend upon the shape of the edge of the gate and upon the actual arrangement of the gate.

The value of the coefficients of discharge of gates varies from about 0.65 for a gate having a square edge to about 0.90 for a gate having no contraction at the sides or bottom, and having a well rounded edge. For a gate to be used as a reasonably accurate measuring device, it would be necessary for it to be calibrated. The head would be read in the same way as for the submerged orifice.

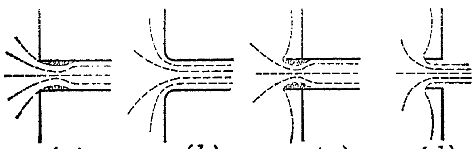
PROBLEMS

107. Find the diameter of a sharp-edged submerged orifice to discharge 4 c.f.s. under a head of 5 ft.

108. Find the discharge of a sharp-edged submerged orifice which is 3 in. in diameter when operating under a head of 3.6 ft. *Ans.* $Q = 0.447$ c.f.s.

109. A gate which extends across a flume 4 ft. in width is raised 1.2 ft. With a head on the gate of 6 ft., the measured discharge was 75 c.f.s. Find the coefficient of discharge for the gate.

62. Flow through Tubes. — At this point, a tube will be defined as a short pipe having a length of two to three pipe diameters. The entrance to the tube may be square, rounded, or re-entrant. The rate of discharge



(a) $C_v = 0.82$, $C_c = 1.00$ (b) $C_v = 0.96$, $C_c = 1.00$ (c) $C_v = 0.75$, $C_c = 1.00$ (d) $C_v = 0.98$, $C_c = 0.52$

FIG. 85. Coefficients for various tubes.

is dependent upon the type of entrance both from the standpoint of the velocity in the jet and also its cross-sectional area. The three types of tubes together with their average coefficients are shown in Fig. 85.

The coefficient of the Borda's mouthpiece, which is a short re-entrant tube, can be derived by the use of the principle of conservation of momentum for cases where the jet does not expand and fill the outlet of the tube. Remembering that the velocity in the jet at the vena contracta is given by the expression $V_2 = C_v\sqrt{2gh}$ and using the expression that impulse equals the change in momentum, we have for a one-second interval

$$F = \frac{wQV_2}{g}$$

where the unbalanced force is whA. We then have

$$whA = \frac{wQV_2}{g} = \frac{wA_2C_v^2\,2gh}{g}$$

or

$$A_2 = \frac{0.5A}{C_v^2} \qquad (101)$$

where A = area of the tube,

A_2 = area of the jet at the vena contracta,

C_v = coefficient of velocity.

From reference to Table V, it is evident that C_v has a mean value in the neighborhood of 0.98. Substituting this value into Eq. (101), we find that the coefficient of contraction should equal 0.52, and except for the introduction of the value of C_v, this value has been obtained without resort to experimental tests. These values correspond to a coefficient of discharge of 0.51. Smith and Walker[1] tested a Borda mouthpiece which was 1 in. in diameter with heads ranging from 10 to 100 ft. and found C_d ranging from 0.510 to 0.525. The re-entrant tube is equivalent to the Borda mouthpiece whenever the jet does not expand and fill the outlet of the tube. The coefficients show a remarkably close agreement between theory and fact.

The jets from the different tubes normally expand and fill the tube at the outlet. The coefficients given on Fig. 85 then apply. With these coefficients, the heads lost in each tube are respectively 0.49, 0.085, and $0.78V^2/2g$. The loss for the bell-mouthed entrance will vary considerably depending upon the perfection of the rounded entrance. In problems involving the flow through pipes, the loss coefficients for the re-entrant and square entrances are normally taken as $V^2/2g$ and $0.5V^2/2g$ respectively. It is understood that the head must be kept low enough so that the absolute pressure at the contracted section does not become equal to the vapor pressure of the liquid.

[1] Smith and Walker, *op. cit.*

63. Flow through Nozzles. — The nozzle to be discussed here may be defined as a gradually converging section attached to the end of a pipe or hose line, the purpose of which is to increase the velocity of the issuing jet. There are other types of nozzles, such as the nozzles on a steam turbine or a measuring nozzle within a pipeline. The nozzle under discussion is illustrated in Fig. 86. The converging portion of the nozzle may be composed of smooth curves or a conical section which ends in a short cylindrical tip. Both types of nozzles have comparable characteristics and the conical nozzle is somewhat preferable due to the simplicity of manufacture. After testing a large number of nozzles, Freeman[1] found the following mean coefficients of discharge for smooth nozzles:

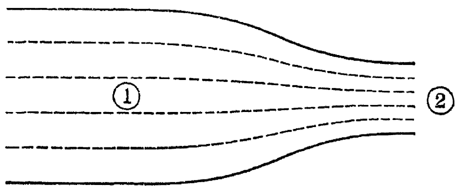


FIG. 86. A simple nozzle.

Nominal diameter, inches	0.75	0.875	1.0	1.125	1.25	1.375
Coefficient	0.983	0.982	0.972	0.976	0.971	0.959[2]

These nozzles were attached to a play-pipe whose inside diameter was 1.55 in. The play-pipe is the short length of pipe to which the nozzle is attached in order that the jet may be directed as desired.

Since the jet diameter from a well designed nozzle is the same as that of the opening, the coefficient of discharge is numerically equal to the coefficient of velocity. The discharge through the nozzle is found by the application of the Bernoulli equation between points (1) and (2) of Fig. 86. The method will be illustrated by means of the following example.

Illustrative Problem: Water flows from a 2-in. pipeline through a $1\frac{1}{8}$ in. nozzle. The pressure at the base of the nozzle is 80 lb. per sq. in.

Find the discharge from the nozzle and the value of the lost head.

Solution: The conditions are identical to those shown in Fig. 86. Writing Bernoulli's equation between points (1) and (2), we have

$$\frac{80 \times 144}{62.4} + 0 + \frac{V_1^2}{2g} = 0 + 0 + \frac{V_2^2}{2g} + \left(\frac{1}{0.976^2} - 1\right)\frac{V_2^2}{2g}$$

but

$$V_1A_1 = V_2A_2, \text{ so } V_1 = \frac{81}{256}V_2$$

[1] Freeman, John R., "Experiments Relating to Hydraulics of Fire Streams." *T.A.S.C.E.*, V. 21, pp. 303–482, 1889.

[2] The throat of the play-pipe was too small for a nozzle of this size, hence low coefficient.

Substituting,

$$184.6 + 0.1\frac{V_2^2}{2g} = \frac{V_2^2}{2g} + 0.05\frac{V_2^2}{2g}$$

$$\frac{V_2^2}{2g} = \frac{184.6}{0.95} = 194.3$$

$$V_2 = 111.8 \text{ ft. per sec.}$$

$$Q = A_2V_2 = (0.0069)(111.8) = 0.77 \text{ c.f.s.}$$

$$\text{The friction loss} = 0.05\frac{V_2^2}{2g}$$

$$= (0.05)(194.3) = 9.72 \text{ ft.}$$

The efficiency of a nozzle is of especial interest in power work, and is defined as the ratio of the energy of the water leaving the nozzle to that which it possessed upon entering. Since the quantity at the inlet and outlet is the same, this can be found by taking the ratio of the total heads at the two points. Referring to the Bernoulli equation which was used in the preceding example, it is evident that

$$H_1 = H_2 + \left(\frac{1}{C_v^2} - 1\right) H_2$$

$$= \left(\frac{1}{C_v^2}\right) H_2$$

$$\text{Efficiency} = \frac{H_2}{H_1} = \frac{H_2}{\left(\frac{1}{C_v^2}\right) H_2} = C_v^2 \qquad (102)$$

PROBLEMS

110. Assuming the discharge through a nozzle to be unity, find the discharge for the same head and diameter from (*a*) bell-mouthed short tube, (*b*) standard orifice, (*c*) Borda mouthpiece. Use reasonable coefficients.

111. Water flows through a short tube 2 in. in diameter which has a square entrance. The vapor pressure of the water for this condition is 5 in. of mercury and the barometer reads 29.6 in. Find the maximum head that can be used on this tube without a tendency for the pressure to go below the vapor pressure.

112. Water flows through a short tube 1 in. in diameter which has a square entrance. The head is 10 ft. Assume that $C_v = 0.85$, $C_c = 1.0$ and that 90 per cent of the lost head takes place between the reduced section and the outlet, find (*a*) the discharge and (*b*) the pressure head at the reduced section.

Ans. (*a*) $Q = 0.118$ c.f.s., (*b*) $p/w = -9.42$ ft.

113. Water flows from a 4 in. pipe through a 2 in. nozzle. The pressure at the base of the nozzle is 150 lb. per sq. in. Find the rate of discharge and the head lost in the nozzle.

114. Find the horsepower delivered by the nozzle of Prob. 113, and the efficiency of the nozzle.

Ans. (*a*) 132 h.p., (*b*) 96 per cent.

115. Find the head required to obtain a discharge of 6 c.f.s. from a 1 in. nozzle on a 1.55 in. line.

116. Find the diameter of a nozzle to discharge 2 c.f.s. of water with the pressure in a 4 in. pipe at the base equal to 40 lb. per sq. in. $C_v = 0.97$.

64. Flow through Venturi Tubes.—The venturi tube, which was invented by Clemens Herschel, consists of a short constriction between two tapered sections and is usually inserted in a pipeline between two flanges. A venturi tube is shown diagrammatically in Fig. 87. The purpose of the constriction is to cause an increased velocity at that point and a corre-

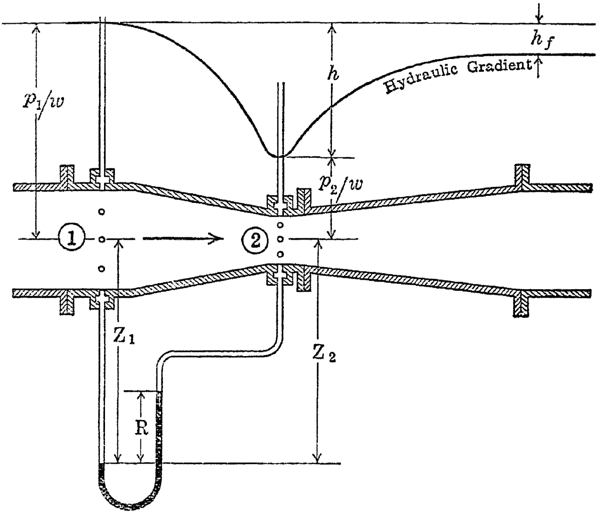


FIG. 87. The venturi tube.

sponding reduction in pressure relative to the pressure upstream from the constriction. The tube consists of a straight portion, of the same diameter as that of the pipeline, which is either cast or machined smooth. The straight portion is connected by a smooth curve to the tapered section whose included angle is about 21°. This section is in turn connected by a smooth curve to the carefully machined cylindrical throat. Connection is then made to the pipe by means of another tapered section whose included angle is between 5° and 7°. The measuring ability of the tube does not depend upon the properties of the expanding section. It is simply the purpose of this section to make connection with the pipeline, and to recover a considerable portion of the pressure differential between the inlet and throat. For well designed venturi tubes, the overall pressure loss in the tube is about 10 to 20 per cent of the pressure differential measured between the inlet and throat.

Small venturis are usually made of brass, or bronze, and made smooth throughout their entire length; while sizes larger than about a 2 in. pipe size are made of cast iron with a brass, or bronze, lining at the throat which is carefully machined. The straight entrance portion may have a smooth lining. Very large venturis with diameters up to 10 ft. have been made of concrete with a machined metal throat.

Provision is made by means of an opening in the side, or by several openings leading to a piezometer ring, for measuring the difference in pressure head between the inlet and the throat. If we write Bernoulli's equation between points (1) and (2) with the losses neglected, we obtain

$$\frac{p_1}{w} + Z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + Z_2 + \frac{V_2^2}{2g}$$

Transposing terms, we obtain

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \frac{p_1}{w} - \frac{p_2}{w} + Z_1 - Z_2 \tag{103}$$

Considering the manometer attached to the meter and letting the datum plane be taken at the lower leg of the gage fluid, we have

$$\frac{p_1}{w} + Z_1 - R\,(\text{S. G.}) - (Z_2 - R) = \frac{p_2}{w}$$

Where S.G. is the specific gravity of the gage fluid with reference to the medium flowing through the meter. Rearranging terms, we obtain

$$\frac{p_1}{w} - \frac{p_2}{w} + Z_1 - Z_2 = R(\text{S.G.} - 1) = h$$

It is now evident that the right hand side of Eq. (103) is equal to the head on the meter and that the manometer will indicate the correct head regardless of the value of the elevation terms. In other words, the meter need not be in a horizontal position. This statement is made with the understanding that the pipes leading to the manometer are filled with the same medium as is flowing through the tube.

Eliminating V_1 from Eq. (103) by means of the equation of continuity, $\pi d_1^2 V_1/4 = \pi d_2^2 V_2/4$, we have

$$\frac{V_2^2}{2g}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right] = h$$

Solving for V_2, we obtain

$$V_2 = \frac{1}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}}\sqrt{2gh}$$

Due to the fact that flow cannot exist without loss, a coefficient must be introduced.

$$V_2 = \frac{C}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}}\sqrt{2gh} \tag{104}$$

and

$$Q = \frac{CA_2}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}}\sqrt{2gh} = \frac{CA_2}{\sqrt{1 - \beta^4}}\sqrt{2gh} \tag{105}$$

where β is the ratio of the two diameters. For a given tube, Eq. (105) can be written

$$Q = K\sqrt{h} \tag{106}$$

The venturi tube and the pipe orifice, which will shortly be discussed, are widely used in industry. Both are suitable for a recording instrument which will either furnish a record of the discharge or will integrate it and thus furnish a record of the total flow. The venturi tube is rather expensive and has the disadvantage that the throat diameter must remain constant for a given installation. This assumes importance when the rate of flow varies between wide limits as the venturi tube does not give accurate results for small differential head readings. The venturi tube offers a continuous head loss in the line but this loss is smaller than would be introduced by the pipe orifice.

65. Coefficients of Venturi Tubes. — The discharge coefficient of venturi tubes depends upon the size of tube and the various factors which appear in the Reynolds number. In other words, a particular tube can be calibrated throughout the range of Reynolds number using a given fluid. This calibration curve may then be used for the same tube while carrying other fluids or for other tubes of the same size and of comparable condition.

A series of tubes were calibrated so the results could be used in a report by the A.S.M.E.[1] in which calibration curves were given for sizes ranging from $\frac{1}{2} \times \frac{1}{4}$ in. up to 200×100 in. Only diameter ratios of 0.5 appeared on the chart, but an empirical equation was given by which the coefficient

[1] Tests on forty-seven tubes from the Simplex Valve and Meter Co. and four from the Builders Iron Foundry were conducted by Prof. W. S. Pardoe with water as the medium. For values below $\mathbf{R} = 10{,}000$, the data was furnished by Ed. S. Smith from tests in which oil was used. The information was used in "Fluid Meters — Their Theory and Applications," *A.S.M.E. Research Publication*, 4th Ed., 1937.

for a tube of any other diameter ratio might be found. Figure 88 has been obtained from this source and Fig. 89 has been plotted from Prof. Pardoe's equation

$$C_\beta = \left[\frac{1-\beta^4}{(1-\beta^4)+\left(\frac{0.9375}{C_{0.5}^2}-0.9375\right)}\right]^{\frac{1}{2}} \tag{107}$$

in which C_β = discharge coefficient for the tube having a diameter ratio β.

$C_{0.5}$ = discharge coefficient for a tube of diameter ratio 0.5 and having the same diameter of throat.

Equation (107) applies for values of β ranging from 0.30 to 0.75 inclusive.

It should be noted that the Reynolds number which appears in Fig. 88 is dependent upon the unknown velocity V_2, but this does not offer any serious disadvantage since a trial value of V_2 can be computed by means of Eq. (104) in which a coefficient of unity has been used. An approximate value of Reynolds number can then be computed and the value of C obtained from Fig. 88. With this value of C, the discharge is computed and, except for values of C which are obtained from the steep portion of the curve, it would rarely be necessary to make a third computation.

Illustrative Problem: Find the discharge of water through an 8 × 5 in. venturi tube. The mercury manometer reads 15 in. and the temperature is 70° F.

$$h = \frac{15}{12}(13.56-1) = 15.7 \text{ ft.}$$

By Eq. (104),

$$V_2 = \frac{}{\sqrt{1-\left(\frac{25}{64}\right)^2}}$$

$$= 34.8 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{D_2V_2}{\nu} = \frac{(5)(34.8)}{(12)(0.000011)} = 1{,}320{,}000$$

From Fig. 88, the coefficient for a 10 × 5 in. tube is found to be 0.983. By use of Fig. 89, the corrected coefficient for a value of β = 0.625 is found to be 0.980. By means of Eq. (105), the discharge is

$$Q = \frac{(0.980)(0.136)}{0.915}\sqrt{2g\,(15.7)}$$

$$= 4.63 \text{ c.f.s.}$$

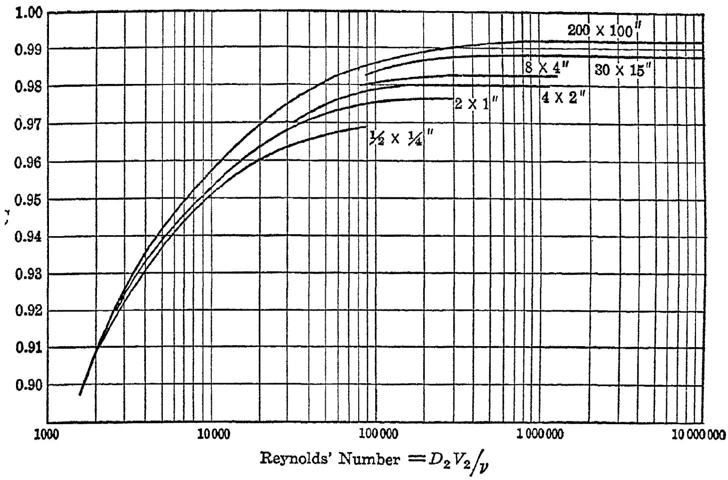


FIG. 88. Coefficients of venturi tubes having a diameter ratio of 0.5.

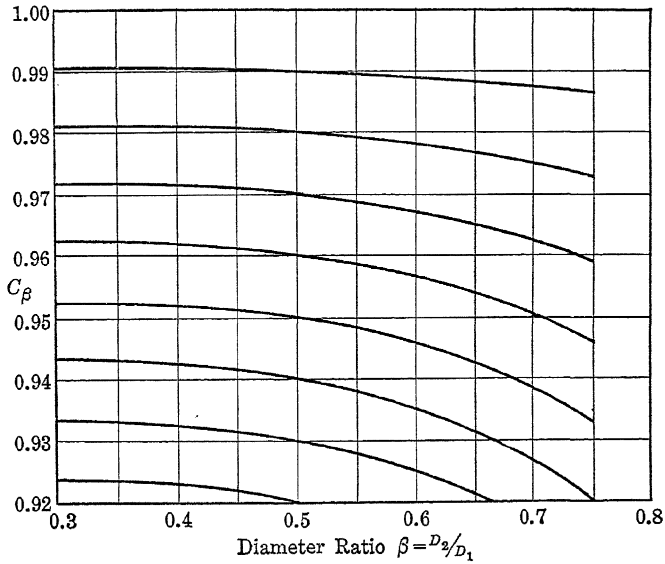


FIG. 89. Variation of venturi tube coefficients with diameter ratio for constant throat size and constant Reynolds number.

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117. Water at 50° F. flows through a 12 × 6 in. venturi tube at the rate of 4 c.f.s. Find the differential head and the reading on a water-mercury manometer.

118. For the same manometer reading as Prob. 117, find the discharge of the 12 × 6 in. venturi when discharging water at a temperature of 160° F.

119. Oil having a kinematic viscosity of 80×10^{-4} sq. ft. per sec. flows through a 12 × 4 in. venturi tube with a differential head reading of 30 ft. of oil. Find the discharge. *Ans.* $Q = 3.48$ c.f.s.

120. It is desired to choose an 8 in. venturi tube such that the differential head will be 16 ft. when water at 75° F. is flowing at the rate of 5 c.f.s. Find the throat diameter. *Ans.* $d = 5.16$ in.

121. Oil having a kinematic viscosity of 45×10^{-5} sq. ft. per sec. and weighing 58 lb. per cu. ft. flows through a 4 × 2 in. venturi tube at the rate of 0.64 c.f.s. Find the head reading on an oil-mercury manometer.

66. Flow through Orifice Meters. — An orifice which is made from a thin plate and inserted in a pipeline between two flanges constitutes an orifice meter. The orifice meter is convenient, simple, and quite accurate when used within the limits which will be stated. While the meter itself is considerably less expensive than a venturi tube, it has the disadvantage

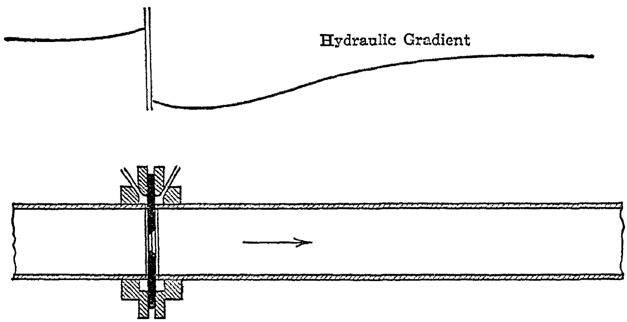

FIG. 90. Variations in pressure caused by orifice meter.

of introducing a greater head loss in the line. The orifice meter is quite adaptable for wide ranges in discharge due to the fact that it can be equipped with a series of orifice plates having a range in the diameter of the opening. These plates can then be interchanged for the different rates of discharge. In this way, a desirable differential head may be maintained.

A thin-plate orifice meter is shown in Fig. 90. There are several positions which are standard for locating the taps (piezometer connections). One of three different taps is normally used in the United States. These will now be described:

Flange Taps. The centers of the pressure holes are located 1 in. from the nearer face of the orifice plate irrespective of the diameter of the pipe. This results in the opening being placed in a different geometric position

for each different diameter of pipe. This is especially poor for orifice meters in small pipes as there is a possibility of the pressure being taken downstream from the point of minimum pressure. The objection of the geometrically changing position assumes less importance as the diameter of the pipe is made larger.

Vena Contracta Taps. For these taps, the centers of the pressure openings are located about one pipe diameter from the upstream face of the orifice plate; while the center of the outlet pressure tap is located in the plane of minimum pressure. The position of the minimum pressure varies with the ratio of the diameter of the orifice to the diameter of the pipe and with the velocity of flow. Where more than one size opening is to be used with one meter, the downstream tap should be placed at one-half pipe diameter from the orifice plate.

The vena contracta taps have the advantage of being located at geometrically similar positions for different pipe sizes.

Pipe Taps. The downstream pressure connection for pipe taps is placed at approximately the section of maximum pressure recovery. In some cases, the opening is downstream from the point of maximum pressure recovery and is thus influenced by the pipe loss itself. The most common combination of distances is to place the upstream tap two and one-half pipe diameters from the upstream face of the orifice plate, and downstream tap eight pipe diameters from the upstream face. These positions require the use of a long measuring device and the measurement will be dependent upon a smaller pressure difference than will be the case with other systems of connections.

Corner Taps. The pressures are taken from the corners formed by the pipe wall and the orifice plate. This can best be accomplished by a narrow circumferential slit between the end of the pipe and orifice plate. These slits can then lead to a recess in the flange to which the pressure connection can be made. This is a very desirable system of connection and one which can be easily duplicated.

Either the corner taps or the vena contracta taps are recommended. Reference to the graph of the pressure variation (known as the hydraulic gradient) appearing in Fig. 90 shows that the differential head is essentially the same for these two types of connections. The corner tap is preferable due to the ease with which it can be manufactured and inspected while being manufactured. The term vena contracta as it is used here is somewhat of a misnomer. Recent tests by O. L. Kowalke[1] show that there is no position of a definitely reduced section on the downstream side of the orifice plate. However, there is a point at which the hydraulic gradient

[1] Kowalke, O. L., "Manner of Liquid Flow through a Pipe-Line Orifice," *Ind. and Eng. Chem.*, V. 30, pp. 216-22, 1938.

shows a minimum. It is this point of minimum pressure that is designated as the vena contracta.

After a comprehensive study, Tuve and Sprenkle[1] recommend for accurate work:

1. The orifice edge must be sharp.
2. The cylindrical portion of the orifice hole shall not exceed one twenty-fifth of the orifice hole diameter.
3. Diameter of piezometer connections for vena contracta taps shall be not less than $\frac{3}{16}$ in. in diameter with the straight hole section at least twice this minimum.
4. Orifice plate shall be perfectly flat and free of all blemishes.
5. Pipe bore shall be smooth and concentric for five diameters preceding and two and one-half diameters following the orifice.
6. Width of the annular recess for corner connections shall be not more than $\frac{1}{8}$ in. for 6 in. pipes and not less than $\frac{1}{32}$ in. for 1 in. pipes.
7. Flange faces for both types of connections shall be flush with the end of the pipe.
8. Flanges and orifice shall be doweled together to insure concentricity of the entire system.

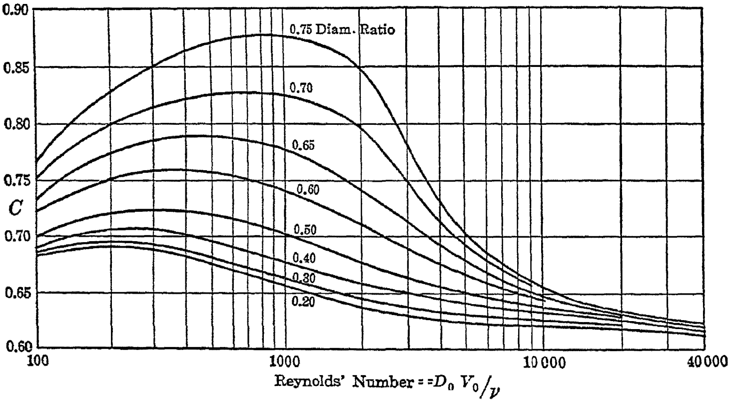

FIG. 91. Variation of coefficient of discharge of orifice meter for different Reynolds numbers.

From data on water and oils, whose viscosity varied from about 1 to 1600 centipoise, Tuve and Sprenkle obtained average coefficients which would apply with an accuracy of $\pm 1\frac{1}{2}$ per cent. These coefficients are given in Fig. 91. The fundamental equation for the orifice meter is the

[1] Tuve, G. L. and Sprenkle, R. E., "Orifice Discharge Coefficients for Viscous Liquids," *Instruments*, V. 6, pp. 201–6; V. 8, pp. 202–5, 225 and 232–4, 1933 and 1935.

same as that of the venturi tube, namely

$$Q = \frac{CA}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}} \sqrt{2gh} \tag{105}$$

The use of the orifice meter is not recommended for values of Reynolds number below 100. The use of large diameter ratios is not recommended for a range of Reynolds numbers near 4000 due to the steep slope of the coefficient curves.

The flow nozzle is somewhat comparable to the orifice meter in that the nozzle is held in the pipeline between pipe flanges. Nozzles of a number of different shapes have been constructed, but the I.S.A. (International Standards Association) nozzle is probably the most standard. An I.S.A. nozzle is shown in Fig. 92. The coefficient of the flow nozzle is more sensitive to variations in the set-up, and to deposits from the fluid than the orifice. This fact is well brought out by Bean and Beitler.[1] The coefficients of the flow nozzle, in general, are somewhat lower than those for the venturi tube. The lost head due to the presence of the nozzle is greater than that caused by the venturi.

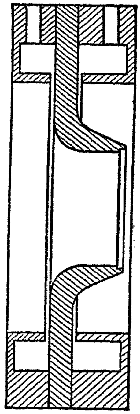

FIG. 92. The I.S.A. Nozzle.

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122. Oil having a kinematic viscosity of 45×10^{-6} sq. ft. per sec. and a specific gravity of 0.93 flows through a thin plate orifice which is mounted in a 3 in. line. The orifice is 1.8 in. in diameter, and the pressure drop across it is 4 lb. per sq. in. Find the discharge. *Ans.* $Q = 0.295$ c.f.s.

123. Water at 60° F. flows through the orifice described in Prob. 122 with the same pressure drop. Find the discharge.

124. Air at a temperature of 80° F. and a pressure of 40 lb. per sq. in. abs. flows through a 2 in. thin plate orifice in a 3 in. diameter pipe. The pressure drop across the orifice is 6 in. of water. Neglecting compressibility, find the discharge through the orifice in pounds per second. Use $\nu = 0.0001$ sq. ft. per sec. and $R = 53.34$.

125. Find the deflection indicated on a water-mercury differential manometer connected across a 4 in. thin plate orifice in an 8 in. pipe which is discharging water at the rate of 0.75 c.f.s. The temperature of the water is 68° F.

67. Weirs. — A *weir* may be defined as an impervious obstruction in a channel over which, or through a notch of which, liquid flows. In America, the term *weir* is normally used to designate a device the principal purpose of

[1] Bean, H. S. and Beitler, S. R., "Some Results from Research on Flow Nozzles," *T.A.S.M.E.*, V. 60, pp. 235–44, 1938.

which is to measure the flow of the liquid in question. In Europe, any overflow type of dam is termed a weir whether or not it is considered to be a measuring device.

Weirs are classified in two ways: either with regard to the shape of the notch when viewed in the direction of flow, or in regard to the shape of the cross section of the dam when viewed normal to the direction of flow. With reference to the first of these methods, one might have a *rectangular*, *triangular*, *trapezoidal*, *circular*, etc.; while with reference to the second, there would be a *sharp crested*, *broad crested*, *ogee*, etc. A rectangular weir is shown in Fig. 93. The triangular weir and the rectangular weir, having a level crest, are the forms most commonly used.

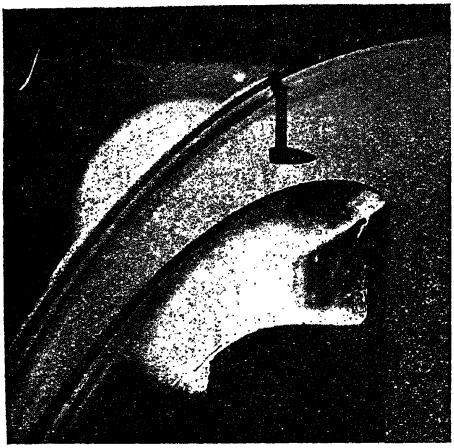

FIG. 93. Flow over rectangular suppressed weir.

As the liquid passes over a rectangular sharp-crested weir, the sheet of liquid is contracted upward due to the vertical component of the velocity along the upstream face of the bulkhead. This sheet of liquid which passes over the weir is called the *nappe*. There is a lateral contraction at the ends of the crest for cases in which the crest does not extend for the full width of the approach channel. There are no end contractions for rectangular weirs having a crest length equal to the width of the approach channel. Such weirs have the end contraction eliminated, or suppressed, and the weir is known as a rectangular suppressed weir. There is also a contraction of the upper surface of the nappe due to the drop down curve. The bottom and end contractions will not be complete when the elevation of the crest of the weir is too near the bottom of the approach channel, or when the vertical edge of the notch is too near the side walls of the approach channel. The bottom contraction will also be decreased when the space below the nappe is not fully ventilated. It is very essential that air have free access to this space. The flow over a weir will always be greater when the contractions are not complete than when they are complete.

The mean velocity in the approach channel is known as the *velocity of approach*. The velocity of approach must be considered when the discharge is being computed.

The height of the liquid surface above the crest of the weir is known as the *head* on the weir. The head must be measured far enough upstream so as to be unaffected by the drop down curve. For the heads normally encountered, the effect of the drop down curve does not extend upstream from the crest more than two or three times the head. The head on the weir must be measured carefully. It is common practice for the head to be measured by means of a hook gage such as is shown in Fig. 94. The hook gage consists of a graduated rod to which is attached a pointed hook. The reading is taken in a stilling basin which is connected to the approach channel by means of a flush piezometer connection. The reading is obtained by first lowering the point below the liquid surface and then raising it by means of a screw until it is seen to touch the surface. The scale reading is read by means of a vernier to thousandths, or ten-thousandths of a foot. The head is then obtained after correcting for the zero of the gage.

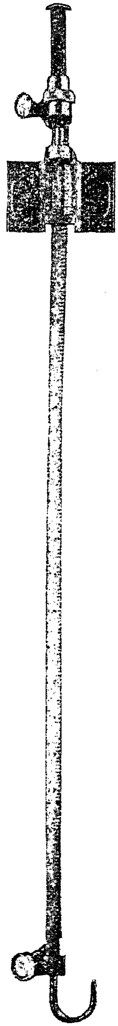
Fig. 94. The hook gage.

68. Rectangular Sharp-crested Weirs. — A sharp-crested weir is one having a square upstream corner such that the liquid which flows over touches it only in a line. The width of the crest parallel to the direction of flow must not be sufficient for the liquid to again come in contact with it.

In obtaining the equation for the discharge over a weir, it is normally assumed that there is no contraction of the nappe and that the velocity of the liquid is equal to that which would be attained by a body falling from rest through the same height y. Neither of these assumptions is even approximately correct as reference to Fig. 95 will show. The values shown were measured by Cox[1] on a rectangular sharp-crested weir 2 ft. long.

The discharge equation will now be obtained for the rectangular suppressed weir using the nomenclature shown on Fig. 96.

$$V_y = \sqrt{2gy}$$

$$dQ = CV_y dA = CV_y L dy = CL\sqrt{2g}y^{1/2}dy$$

$$Q = CL\sqrt{2g}\int_{\alpha\frac{V^2}{2g}}^{H+\alpha\frac{V^2}{2g}} y^{1/2}dy = C\frac{2}{3}L\sqrt{2g}\left[y^{3/2}\right]_{\alpha\frac{V^2}{2g}}^{H+\alpha\frac{V^2}{2g}}$$

$$Q = C\frac{2}{3}L\sqrt{2g}\left[\left(H+\alpha\frac{V^2}{2g}\right)^{3/2} - \left(\alpha\frac{V^2}{2g}\right)^{3/2}\right] \qquad (108)$$

[1] Cox, Glen N., "The Flow of Water Over a Rectangular Weir as Affected by Various Degrees of Roughening of Its Upstream Face," an unpublished thesis, *Univ. of Iowa*, 1926.

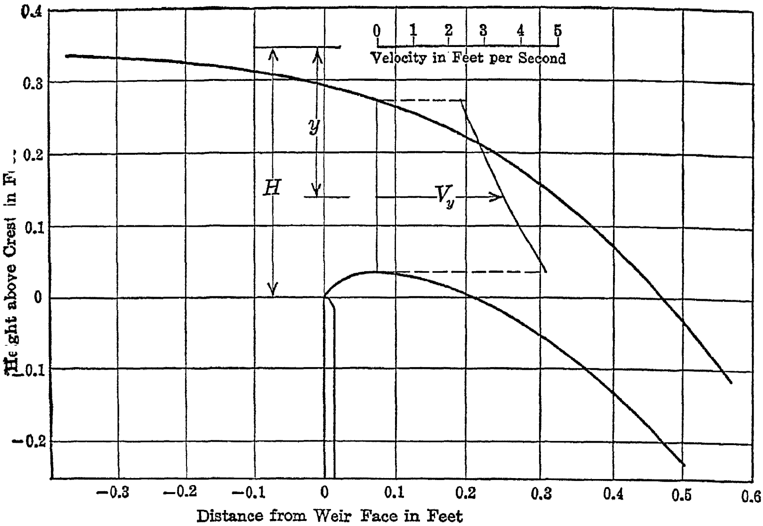

FIG. 95. Profile of nappe and velocity distribution for rectangular suppressed weir.

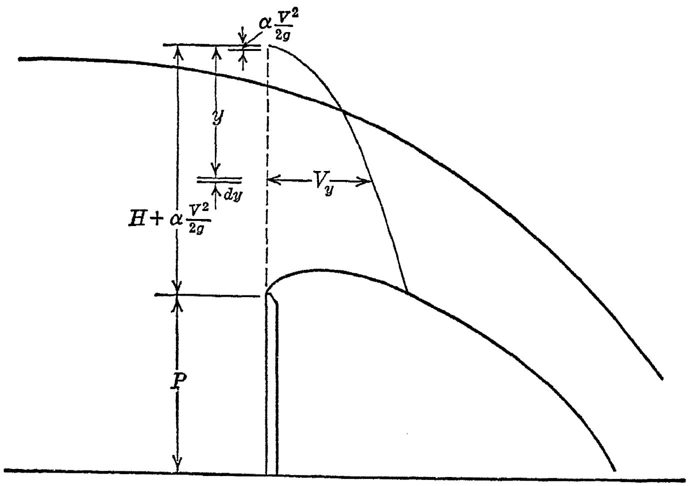

FIG. 96. Assumed velocity distribution for rectangular suppressed weir.

in which Q = discharge in c.f.s.,
H = head on the weir in feet,
V = mean velocity in the approach channel in feet per second,
α = coefficient for obtaining effective velocity of approach head,
C = weir coefficient of discharge,
L = length of weir crest in feet.

According to Smith,[1] the value of α varies between the limits of 1.33 and 1.4. The value[2] of C varies as shown in Fig. 97. The reader should note the similarity of the coefficients for the sharp-crested weir and for the

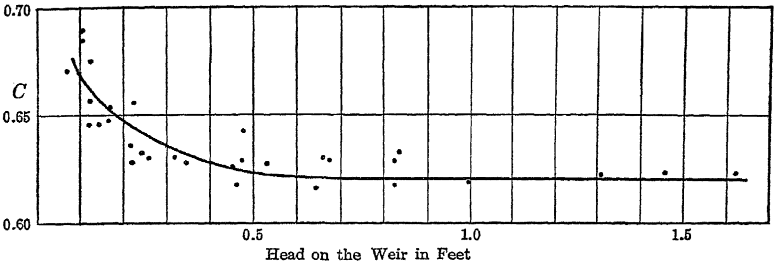

FIG. 97. Variation in rectangular weir coefficient for changes in head.

standard orifice. Should the discharge over a suppressed weir be computed without considering the velocity of approach correction, the term within the brackets of Eq. (108) would reduce to $H^{3/2}$.

Should we recognize the fact that the flow does not take place through an area of LH, but through an area of approximately $0.67LH$ which extends from $0.22H$ to $0.89H$ as shown in Fig. 95, an entirely different value would have been obtained for the coefficient of the weir. Neglecting the velocity of approach correction, the expression for the discharge would become

$$Q = C'L\sqrt{2g}\int_{0.22H}^{0.89H} y^{1/2}dy$$

$$= C'\frac{2}{3}L\sqrt{2g}\,(0.74H^{3/2})$$

Considering an average of 0.62 for the value of C, the corresponding value of C' would be 0.84. While C' would be a more logical coefficient, custom dictates the use of the coefficient C.

[1] Smith, Hamilton, " Hydraulics," John Wiley and Sons, 1886.
[2] *Ibid.* pp. 95 and 96.

The value of the weir coefficient C is determined by experiment. From the results of tests conducted by many investigators, a number of empirical equations have been developed. These equations of necessity reflect the characteristics of the particular apparatus with which the test results were obtained and, in general, the equations should not be used for obtaining the discharge over weirs differing appreciably from the original, or for heads outside the range covered by the equation in question. Certain of the commonly accepted formulas will now be discussed.

Francis Formula. Probably the most used formula is that developed by J. B. Francis[1] from carefully conducted tests on weirs 8 and 10 ft. long with heads ranging from 0.6 to 1.6 ft. His equation for the rectangular suppressed weir is

$$Q = 3.33L\left[\left(H + \frac{V^2}{2g}\right)^{3/2} - \left(\frac{V^2}{2g}\right)^{3/2}\right] \tag{109}$$

Comparing equations (108) and (109), we see that Francis has used values of $C = 0.623$ and $\alpha = 1$. Upon referring to Fig. 97, it is seen that this value of C will give good results for the range of heads tested, but is too low for lower heads.

The value of V is dependent upon the discharge and it is necessary to solve the equation by trial and error. This offers no serious handicap as a trial value of Q can be computed without consideration of the velocity of approach correction. Using a velocity of approach correction based upon the trial value of Q, a new discharge can be computed. It would normally not be necessary to use more than a second trial.

Bazin Formula. H. Bazin[2] conducted a very extensive series of experiments on the flow over weirs of various heights. The water passed over a calibrated measuring weir 1.135 meters (3.72 ft.) high and then over the weir under investigation. Four weirs were used having heights of 0.75 m. (2.46 ft.), 0.50 m. (1.64 ft.), 0.35 m. (1.15 ft.) and 0.24 m. (0.79 ft.). By this method, a considerable range was obtained in the value of the velocity of approach. The heads investigated ranged from 0.09 m. (0.29 ft.) to 0.44 m. (1.44 ft.).

Bazin used Eq. (108) with the small last term omitted for the fundamental equation and then by a series of algebraic transformations and approximations obtained a value of m from the equation

$$Q = mLH\sqrt{2gH}$$

[1] Francis, J. B., "Lowell Hydraulic Experiments," 1883.

[2] Bazin, H., "Recent Experiments on the Flow of Water Over Weirs." Translated by Marichal and Trautwine, *Proc. of Engrs. Club of Philadelphia*, Jan. 1890.

which was expressed by

$$m = \mu\left[1 + K\left(\frac{H}{P + H}\right)^2\right]$$

in which P is the height of the crest above the floor of the channel in feet. From the series of tests upon the weirs of various heights, experimental values of μ and K were determined. The final form of the equation with the constants expressed in English units is

$$Q = \left(3.25 + \frac{0.079}{H}\right)\left[1 + 0.55\left(\frac{H}{P + H}\right)^2\right]LH^{3/2} \qquad (110)$$

Rehbock's Formula. Prof. Rehbock has been studying the flow of water over weirs at Karlsruhe Hydraulic Laboratory, Karlsruhe, Germany, for more than a quarter of a century during which time many different weirs were tested. In 1912[1], Rehbock presented the formula which for English units appears as

$$Q = \left(3.24 + \frac{5.35}{320H - 3} + 0.428\,\frac{H}{P}\right)LH^{3/2} \qquad (111)$$

Equation (111) is not dimensionally correct and was later simplified to the following[1] which is dimensionally correct:

$$Q = \left(3.22 + 0.445\,\frac{H}{P}\right)L(H + 0.004)^{3/2} \qquad (111a)$$

in which the last term has the dimension of a length.

Rehbock does not recommend the use of weirs having a crest height in excess of 4.0 ft. nor for heads less than 0.08 ft. or greater than 2.0 ft. For careful measurements within this range, he claims that his equation will give results which will be within less than 1 per cent of the correct value. It is probable that no other formula will cover this whole range with such accuracy.

Correction for End Contractions. Francis studied the effect of end contractions upon the discharge over rectangular weirs and obtained an effective length of crest by the use of the expression

$$L' = L - 0.1nH \qquad (112)$$

Where L' is the effective length of the crest and n is the number of complete end contractions. The value of n is normally 2. The discharge over the weir is then obtained by substituting this effective length in the equation for the suppressed weir in the place of the actual length.

[1] See discussion by Prof. Rehbock of Schoder and Turner, "Precise Weir Measurements," *T.A.S.C.E.*, V. 93, pp. 1143–62, 1929.

By inspection, it is evident that the value of H must not be great in comparison to L, otherwise the correction will be too great. Smith[1] has recommended the proportions shown in Fig. 98 as minimum values. Should these minimum proportions be used, there would be an increase in discharge of about $\frac{1}{2}$ per cent.

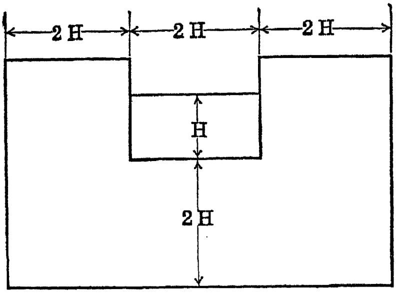

FIG. 98. Minimum proportions for rectangular contracted weir.

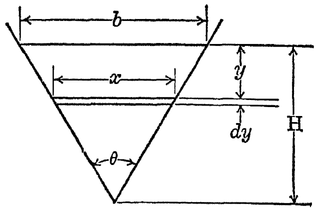

FIG. 99

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126. Water under a head of 0.2 ft. flows over a sharp-crested suppressed weir 3 ft. long whose crest is 2.5 ft. above the bottom of the approach channel. Find the discharge using Eq. (109), (110), and (111).

127. Water under a head of 1.5 ft. flows over a sharp-crested suppressed weir 3 ft. long whose crest is 2.5 ft. above the bottom of the approach channel. Find the discharge using Eq. (109), (110), and (111). *Ans.* Q = 18.9, 19.6, 19.3 c.f.s.

128. Find the head on a sharp-crested suppressed weir 3 ft. high and 4 ft. long when the discharge is 30 c.f.s.

129. Water 2 ft. deep flows in a rectangular channel 8 ft. wide with a mean velocity of 3 ft. per sec. A sharp-crested suppressed weir is placed across the channel and the depth of the water on the upstream side is increased to 4.3 ft. Find the height of the crest.

130. Find the discharge over a rectangular contracted sharp-crested weir 5 ft. long whose crest height is 2.8 ft. when operating under a head of 0.9 ft. The approach channel is 10 ft wide.

69. Triangular Weirs. — The sharp-edged triangular weir is often used for measuring water when the discharge is too low for the successful use of the rectangular weir. The cross-sectional area of the nappe is small for low heads but a small discharge produces a head of such a magnitude on the triangular weir that it can be accurately measured.

A triangular weir having a vertex angle θ is shown in Fig. 99. The discharge through the element of area dA will be

$$dQ = C\sqrt{2gy}\,dA$$

[1] Smith, Hamilton, "Hydraulics," p. 121, John Wiley and Sons, 1886.

but $dA = xdy$; and by similar triangles $x : b = (H - y) : H$. Therefore, $dA = (b/H)(H - y)dy$. Substituting this value of dA in the preceding expression, the following is obtained for the discharge of the weir:

$$Q = C\sqrt{2g}\frac{b}{H}\int_0^H (Hy^{1/2} - y^{3/2})dy$$

Integrating and evaluating, it follows that

$$Q = C\sqrt{2g}\frac{b}{H}(\tfrac{2}{3}H^{5/2} - \tfrac{2}{5}H^{5/2})$$

But $b = 2H \tan \theta/2$. Substituting this and reducing,

$$Q = \frac{8}{15}C\sqrt{2g}\tan\frac{\theta}{2}H^{5/2} \tag{113}$$

For a given weir, Eq. (113) may be reduced to

$$Q = KH^{5/2} \tag{114}$$

Very careful tests on the discharge of V-notch weirs have been made by many experimenters with water as the flowing medium. Notable among these are the tests of Barr,[1] Cone,[2] Yarnall,[3] and Greve[4]. For water flowing, the coefficient varies slightly with the head, with the angle of the notch and with the temperature of the water. Greve found from the test with water of a number of weirs having different angles that the discharge could be expressed within about 1 per cent by the formula

$$Q = 2.5\left(\tan\frac{\theta}{2}\right)^{0.996}H^{2.47} \tag{115}$$

Equation (115) was based on weirs whose angles varied from 25° to 118°. The equation will not apply for angles much less than the 25° due to the greater relative influence of capillarity.

In spite of the variations noted, good average results may be obtained by the use of the following formulas when water is flowing:

$$\theta = 90^\circ \qquad Q = 2.48H^{2.48} \tag{116}$$

$$\theta = 60^\circ \qquad Q = 1.42H^{2.45} \tag{117}$$

[1] Barr, J., "Flow of Water over Triangular Notches," *Engineering*, V. 89, pp. 435–37, 438, 473, 1910.

[2] Cone, V. M., "Flow through Weirs Notches with Thin Edges and Full Contractions," *Jour. Ag. Research*, V. 5, 1916.

[3] Yarnall, D. R., "Accuracy of the V-Notch Weir Method of Measurement," *T.A.S.M.E.*, V. 48, p. 939–64, 1926.

[4] Greve, F. W., "Flow of Water through Circular, Parabolic, and Triangular Vertical Notch-weirs," *Engr. Expt. Sta. Bul.*, No. 40, Purdue University, Lafayette, Ind., 1932.

The discharge of weirs is affected by surface tension and viscosity. Since these properties are not the same for various liquids, a discharge equation which was obtained with water as the medium does not apply for the different liquids. The effect of surface tension is felt more at low heads, but the weir is not a satisfactory measuring device for low heads because of the uncertainty as to whether or not the nappe will spring free from the weir face. The weir must not be used for heads below which the nappe does not spring free. The value of this minimum head is dependent upon the surface tension of the flowing liquid.

The discharge of the weir depends upon the head H, the value of gravity g, the density of the discharging liquid ρ, its viscosity μ and surface tension σ, the relative roughness of the weir plate k, the angle of the weir θ, the height of its vertex p, the width of the approach channel b, and the thickness of the crest w. These conditions are expressed by the functional relationship

$$f_1(Q, H, g, \rho, \mu, \sigma, k, \theta, p, b, w) = 0$$

For any one weir plate in a given weir box, k, θ, p, b and w do not change and f_1 can be replaced by the simpler form

$$f_2(Q, H, g, \rho, \mu, \sigma) = 0$$

Since f_2 contains $n = 6$ quantities containing $i = 3$ fundamental dimensions, there are

$$n - i = 3\ \pi\text{-terms}$$

Let $\qquad \pi_1 = Q^a H^b \rho^c g, \quad \pi_2 = Q^d H^e \rho^f \mu \quad \text{and} \quad \pi_3 = Q^g H^h \rho^i$

Placing $\qquad f_3\left(\sqrt{\frac{1}{\pi_1}}, \ \frac{1}{\pi_2}, \ \frac{1}{\pi_3}\right) = 0$

we obtain $\qquad f_3\left(\frac{Q}{H^{5/2}\, g^{1/2}}, \frac{Q}{H\nu}, \frac{gH^2\rho}{\sigma}\right) = 0$

In f_3, the dimensional equivalents, $Cg^{1/2}H^{5/2}$ will be substituted for the first Q and $g^{1/2}H^{5/2}$ for the second one.

These substitutions[1] have been made in order to obtain a function which contains only measurable quantities, and one which has all of these grouped in two of the π-terms.

[1] Similar substitutions for surface tension neglected were made by H. N. Eaton in his discussion of "The V-Notch Weir for Hot Water" by Ed. S. Smith, *T.A.S.M.E.*, V. 57, p. 249, and by A. T. Lenz for surface tension included in "The Effect of Viscosity and Surface Tension upon V-Notch Weir Coefficients," an unpublished Ph. D. thesis, Univ. of Wis., 1940.

It then follows that

$$C = f_4\left(\frac{g^{1/2}H^{3/2}}{\nu}, \frac{gH^2\rho}{\sigma}\right)$$
$$= f_4(\mathbf{R},\mathbf{W})$$

The value of C has been used in this equation because it is dimensionless, while K in Eq. (114) is not dimensionless. Since for any one locality the value of g does not change, the effect of viscosity and surface tension upon

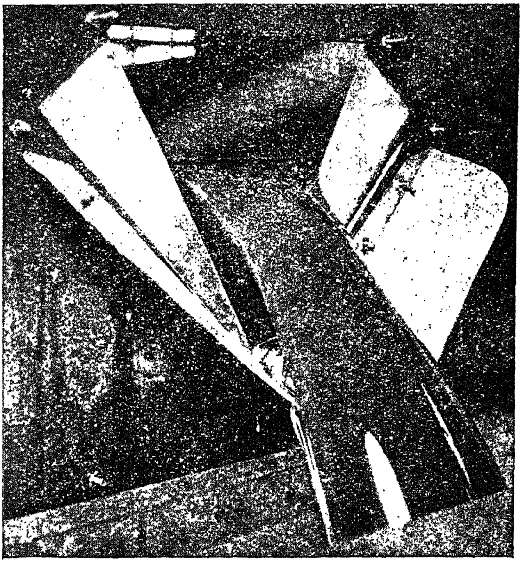


FIG. 100. Flow of oil over 90° notch. (Courtesy of A. T. Lenz.)

the discharge can be found by studying the variation in C for different values of Reynolds' and Weber's numbers. These forms are especially desirable since they are independent of the discharge, thus making it possible to obtain directly the value of the coefficient for a given set of conditions.

A very carefully conducted series of tests on the flow of water, Pennsylvania oil and fuel oil over V-notch weirs was reported by Lenz.[1] The angle of the notch was varied from 10° to 90°. The flow of the Pennsylvania oil over the weir is shown in Fig. 100. Professor Lenz found that the coefficient could be expressed by the equation

$$C = 0.56 + \frac{B}{\mathbf{R}^n\mathbf{W}^m} \tag{118}$$

[1] Lenz, A. T., *op. cit.*

where B, n and m were dependent upon the weir angle and have the values given in Fig. 101. The variation in the viscosity of the liquids tested was in the order of 160 to 1, while that of the surface tension was 2 to 1.

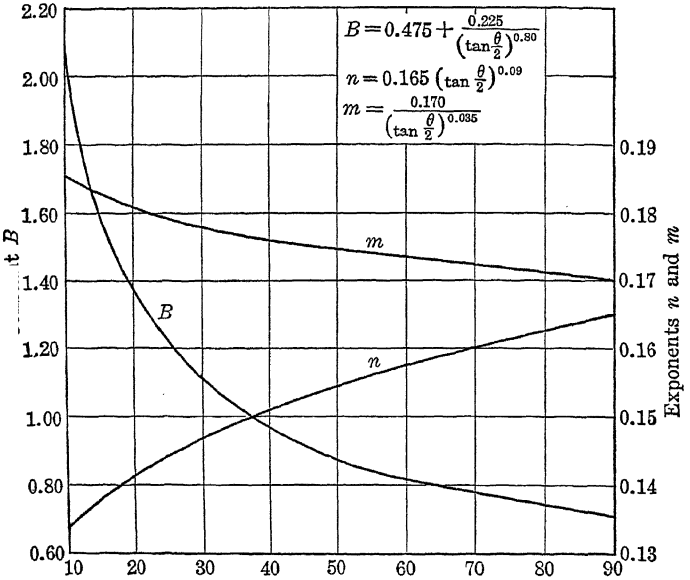


FIG. 101

The values of C obtained by use of Eq. (118) agree with experimental results within about 1 per cent, subject to the following limiting conditions:

1. The minimum Weber's number should not be below 300. This corresponds to a water head of about 0.15 ft., or an oil head of about 0.11 ft.
2. Equation (118) no longer applies when the value of Reynolds number is less than

$$\mathbf{R} = \frac{300}{\left(\tan\frac{\theta}{2}\right)^{3/4}}$$

3. For large values of Reynolds number, the coefficient reaches a minimum constant value of approximately 0.59.
4. While the Wisconsin tests covered a range in angles from 10° to 90° and while good agreement was obtained by use of Eq. (118), other experimenters have found difficulty in obtaining good agreement for small angles. For this reason, it is recommended that the angle be kept in the range from 28° to 90°.

The reader is again reminded that the value of K is not dimensionless. It has the same dimensions as $g^{1/2}$ and equations (116) and (117) presented here are for use with English units. The simplicity of these equations as compared to equations (113) and (118) warrants their use.

PROBLEMS

131. Water at a temperature of 90° F. flows over a 90° V-notch weir under a head of 1.2 ft.

(*a*) Find the discharge using Eq. (115) and Eq. (116).

(*b*) Find the discharge using Eq. (118). $\sigma = 0.00485$ lb. per ft.

132. Find the head required on a 60° V-notch weir to give a water discharge of 1.5 c.f.s.

133. Water flows over a sharp-crested rectangular suppressed weir which is 2 ft. long and then over a 90° V-notch weir. The head is the same on both weirs.

(*a*) Find the head.

(*b*) Find the discharge.

134. Oil having a viscosity of 85 S.S.U. and a surface tension of 0.0021 lb. per ft. flows over a 90° V-notch weir under a head of 0.75 ft. Find the discharge, taking S.G. = 0.93. *Ans.* $Q = 1.223$ c.f.s.

135. Oil having a kinematic viscosity of 0.0004 sq. ft. per sec. and a surface tension of 0.0022 lb. per ft. flows over a 90° V-notch weir at the rate of 0.6 c.f.s. Find the head, if S.G. = 0.86.

136. Water at a temperature of 60° F. flows over a 35° V-notch weir under a head of 0.9 ft. $\sigma = 0.00503$ lb. per ft. Find the discharge.

137. Water flows over a 90° V-notch weir under a head of 0.2 ft. Find the discharge if

(*a*) the temperature of the water is 32° F. Assume $\sigma = 0.00525$ lb. per ft.

(*b*) the temperature of the water is 120° F. Assume $\sigma = 0.00222$ lb. per ft.

Ans. (*a*) 0.0466 c.f.s., (*b*) 0.0455 c.f.s.

70. Trapezoidal Weirs. — The discharge of a trapezoidal weir is usually considered to be the sum of the discharges of a rectangular and a triangular weir. Referring to Fig. 102, it can be seen that the discharge from the triangular portions can be made to compensate for the decrease in discharge due to the end contractions. An Italian engineer, named Cipolletti, proposed a side slope of $\frac{1}{4}$ in order that the coefficient would not change with varying values of head. Cipolletti proposed the formula

$$Q = 3.367LH^{3/2} \tag{119}$$

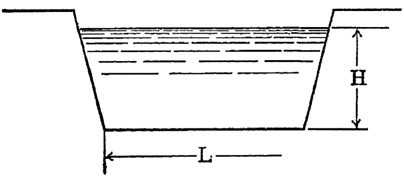


FIG. 102. The trapezoidal weir.

The few tests that have been conducted on the flow over Cipolletti weirs indicate that the coefficient does not remain constant. For this reason, this type of weir is not recommended where a high degree of accuracy is needed.

The Cipolletti weir has found its chief use in the measurement of irrigation water. It is satisfactory for this purpose since a high degree of accuracy cannot be attained with the changing channel conditions.

71. Broad-Crested Weirs. — The broad-crested weir has essentially a rectangular cross-section with a flat crest whose width parallel to the axis of the channel must be about twice the head. Should the width be small, the nappe will spring free at the upstream edge and the discharge will be comparable to that over a sharp-crested weir. A broad-crested weir is shown in Fig. 103.

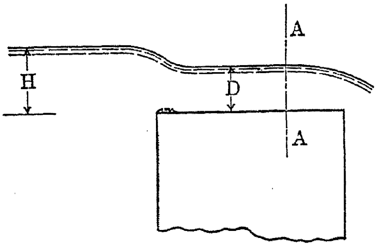


FIG. 103. The broad-crested weir.

The discharge equation for the broad-crested weir will now be obtained. Since the liquid particles move in a horizontal line as they pass through the reduced section, *A-A*, over the weir, the pressure will increase uniformly with the depth. The velocity in this section will then be uniform and neglecting friction, will be equal to

$$V = \sqrt{2g(H - D)} \tag{a}$$

This equation is obtained by writing Bernoulli's equation between a point upstream from the weir and a point at section *A-A*. Then

$$Q = AV = LD\sqrt{2g}\sqrt{H - D}$$

$$= \sqrt{2g}L(D\sqrt{H - D}) \tag{b}$$

The value of D will be such as to cause a maximum discharge. It can readily be seen that $V = 0$ when $D = H$ and that $A = 0$ when $D = 0$. For each of these conditions, there could be no discharge. For some intermediate value of D, the discharge would be a maximum. This value of D can be found by maximizing the term in the parenthesis of (b). The value of D for maximum discharge is found to be $\frac{2}{3}H$. Substituting this value in (b), we obtain

$$Q = \frac{2}{3\sqrt{3}}\sqrt{2g}LH^{3/2}$$

or

$$Q = 3.087LH^{3/2} \tag{c}$$

Due to the fact that losses exist, the discharge is less than that given in (c) and, for a weir having a square upstream corner, can be found by the

equation[1]

$$Q = 2.63LH^{3/2} \tag{120}$$

If the upstream corner is rounded, thus reducing the losses, the value of the coefficient will approach the theoretical maximum value of 3.087 as given above by Eq. (c).

PROBLEMS

138. In Eq. (b), prove that $D = \frac{2}{3}H$ for a maximum discharge.

139. A broad-crested weir which is 20 ft. long has a square upstream corner. Find the discharge when the head is 1.4 ft.

140. Find the head needed to discharge 80 c.f.s. over a broad-crested weir 10 ft. long. *Ans.* $H = 2.10$ ft.

72. Submerged Weirs. — A weir is said to be submerged, see Fig. 105, whenever the tail-water rises to such an elevation that the discharge over the weir is affected. This may occur with the downstream head below the elevation of the crest. Any type of weir may become submerged, but it is probable that the most practical example would be the case of a dam being *drowned* during flood stages. In such cases, it is often desirable to be able to compute the rate of discharge with some degree of certainty. The submerged weir would also make a desirable measuring device to be installed in an open flume whenever the head which could be sacrificed for measuring purposes was small.

Tests on the flow over submerged weirs have been made by a number of experimenters, notably Bazin, Francis, Fteley and Stearns, and Cox, but the agreement of the test results was not considered good. This was due, in large measure, to the fact that the downstream head was measured in a different relative position. As water flows over a submerged weir, a trough is formed downstream from the crest after which the surface rises somewhat. Bazin measured the head at a considerable distance downstream, past the point of recovery. Francis, and Fteley and Stearns measured the downstream head near the bottom of the trough. It is evident that the per cent submergence, which is the ratio of the two heads expressed as a per cent, would be quite different for these positions of measurement. It seems logical that the downstream head should properly be measured in the trough since the magnitude of the dropdown is largely a function of the discharge over the crest, while the recovery which takes place downstream from the trough is essentially a function of the shape of the channel beyond the trough. The recovery is not a function of the flow alone. A further

[1] See *U. S. Geological Survey, Water Supply Paper No.* 200 by R. E. Horton for calibration data of broad-crested weirs and of many other types which are not sharp-crested.

advantage exists from measuring the downstream head in the trough in that the distance to the point of maximum recovery for suppressed weirs is approximately ten times the height of the crest of the weir. For a natural stream, it would be difficult to obtain a uniform channel for that distance.

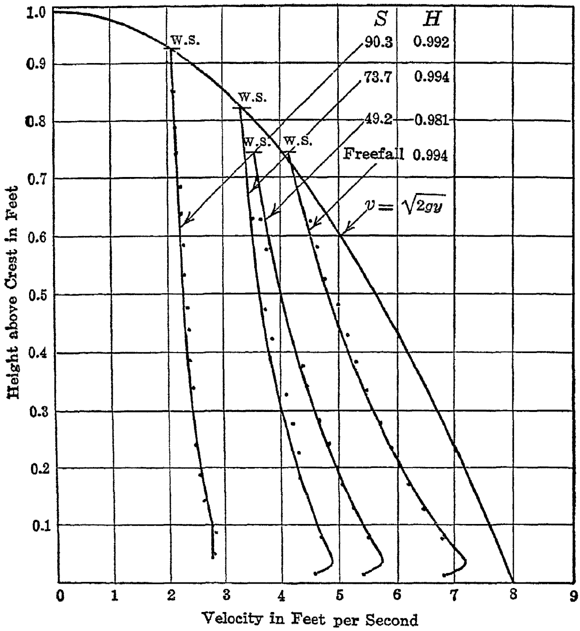


FIG. 104. Velocity profiles for various submergences on submerged weir.

A fundamental error of analysis has persisted with many writers. These writers assume that the discharge is the sum of the discharge over a free-fall weir for the portion above the downstream head elevation, and the discharge from a submerged orifice for the portion between this and the elevation of the crest. Such an assumption would require a velocity profile similar to that for a weir for the upper portion and then a constant velocity for the lower portion. Such a requirement is quite illogical and not true to fact[1] as reference to Fig. 104 clearly indicates. It might better be said that the downstream water merely acts as a resistance, or brake, to the passage of the water over the weir. There is little alteration in the general shape of the velocity profile regardless of the amount of submergence.

[1] Cox, Glen N., " The Submerged Weir as a Measuring Device," Univ. of Wis., Madison, Wis., *Engr. Expt. Sta. Bul.*, No. 67, 1928.

As water passes over the crest of a submerged weir, the nappe either plunges below the surface or flows away on the surface. The condition of the nappe plunging below the surface is shown for an ogee weir in Fig. 105. This type of flow is the more stable for the ogee weir, and the ogee weir is a very satisfactory measuring device for this type of flow.

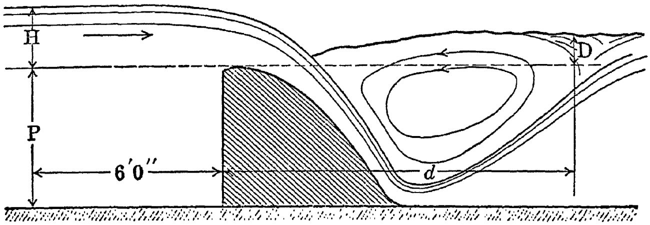


FIG. 105. The ogee submerged weir.

The Wisconsin tests covered sharp-crested weirs, ogee vertical back weirs, and ogee weirs having 2 to 1 upstream faces. The crest heights ranged from 1.14 to 6.11 ft. and the head ranged from 0.102 to 2.147 ft. Due to the relatively greater importance of the vertical back ogee type of weir under these conditions, only the flow over it will be discussed.

For all types of submerged weirs, the general equation

$$Q = CLH^n$$

is applicable. For the ogee weirs, neither C nor n remains constant for different submergences on the same weir or for different heights of crest. For the condition of nappe flowing under, the following empirical expressions were obtained for C and n:

$$C = 3.895 - \frac{0.433S^2}{1.35 - S} \tag{121}$$

$$n = 1.625 - \frac{0.140S^2}{3 - S} \tag{122}$$

Over the range tested, the values of C and n were found to be independent of the crest height. Such a condition did not exist when the nappe remained on the surface, and the equations for C and n contained the crest height as a variable. For nappe over conditions

$$C = 3.895 + 0.01064P - \frac{(1.52 - 0.1886\sqrt{P})S^3}{1.433 - 0.01353P - S} \tag{123}$$

$$n = 1.625 + \frac{0.036}{\sqrt{P}} \quad \frac{\left(\dfrac{0.428}{P - 0.169} - 0.0129\right)}{0.886 + \dfrac{1.31}{P} - S} \tag{124}$$

where P is the height of the weir in feet and S is the submergence expressed as a decimal.

In computing the value of S, a velocity of approach correction must be added to the observed value of the upstream head before the submergence

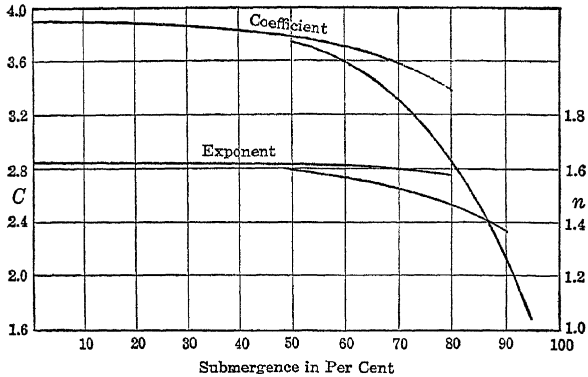


FIG. 106. Coefficient and exponent curves for ogee submerged weir.

is computed. The downstream head must be measured in the trough. The position of the trough is given by the expression

$$d = 2.45P \tag{125}$$

where d is the distance in feet measured from the vertical face of the weir.

While equations (121) to (124) may appear somewhat formidable, coefficient and exponent curves for any one weir can be easily plotted. The computation of the desired values is then not difficult. A sample of such curves is shown in Fig. 106.

Equations (121) to (125) gave results within about ± 3 per cent for submergences below 95 per cent but since they are empirical, an opportunity for testing them for values beyond the range of the tests was welcomed and occurred in 1939.[1] Values were observed on a dam which differed somewhat in shape from the test weir. The dam was 8 ft. high and the maximum head was 6 ft. Over the whole range of heads, the maximum difference between the observed upstream head and the value computed by the use of equations (121) to (124) was 0.16 ft. and occurred for an upstream

[1] Cox, Glen N., "Submerged Weir Formula Verified," *Eng. News-Rec.*, V. 123, pp. 272–3, 1939.

head of 2.4 ft. This difference was less than 7 per cent. The numerical difference between the two heads decreased for further increase in head and amounted to 0.10 ft. with a head of 5.85 ft. This difference amounted to 1.7 per cent.

PROBLEMS

141. Water flows over an ogee vertical back weir having a crest length of 3 ft. and a crest height of 2.13 ft. The measured upstream and downstream heads are 0.938 and 0.554 ft. respectively. The nappe plunges under. Find the discharge.

142. The conditions in Prob. 141 remain unchanged except that the nappe stays on the surface. Find the discharge.

143. Find the upstream head needed if the discharge over an ogee vertical back weir 3 ft. high is 5.80 c.f.s. per ft. of length. The nappe flows over and the downstream head is 1.14 ft. *Ans.* $H = 1.49$ ft.

144. A flood overtopped a railway embankment which was 8 ft. high for a distance of 500 ft. The measured upstream and downstream heads were 5.0 and 4.3 ft. respectively. The nappe remained on the surface.

Compute the flow assuming that the section of the embankment approximates an ogee section.

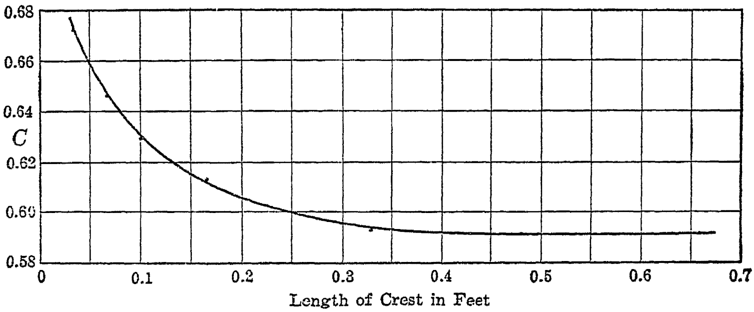


FIG. 107. Coefficient curve for narrow notches.

73. Flow through Narrow Notches. — The minimum crest length for a contracted weir in terms of the head was given in Fig. 98. Weirs having crest lengths less than $2H$ will be designated as narrow notches. Under a given condition, the narrow notch has a relatively greater discharge than the contracted weir since the crest length is not sufficient to permit the full end contraction.

The work of Lebros and Castel as reported by Smith[1] covered tests of narrow notches having crest lengths ranging from 0.033 to 0.654 ft. The maximum head for these tests was 1.947 ft. For heads exceeding 0.13 ft., there were only minor variations in the value of the coefficients for any one crest length. The value of the mean coefficients are shown as a function

[1] Smith, Hamilton, "Hydraulics," pp. 138–139, John Wiley and Sons, 1886.

of the crest length in Fig. 107. These coefficients are to be used in the weir formula

$$Q = C\tfrac{2}{3}L\sqrt{2g}\left(H + \alpha\frac{V^2}{2g}\right)^{3/2} \tag{126}$$

where $\alpha = 1.4$

The attention of the reader is directed to the similarity in the magnitude of these coefficients with those of circular orifices. It is apparent that a perfect contraction cannot be obtained for crest lengths less than about 4 in.

PROBLEM

145. A 1 in. notch whose crest is 4 in. above the floor of the approach channel is placed in a 10 in. channel. Find the discharge when the head is 0.85 ft.

Ans. $Q = 0.223$ c.f.s.

B. COMPRESSIBLE FLUIDS

74. General. — As a compressible fluid flows through a measuring device, it passes from a region of high pressure to one of lower pressure and expands as it passes into the region of lower pressure. Due to this expansion, a given weight of gas will occupy a greater volume than would have been required had the expansion not occurred. As a result, the numerical value of V_2 will be greater with respect to V_1 than it would have been without the expansion, but the product w_2V_2 would be smaller than the product wV_2 for a non-compressible fluid. In other words, due to the expansion of the gas, the change in kinetic energy between the inlet and throat will be greater for a gas than for a liquid. This results in a smaller discharge when compressibility is considered over that which would have existed without compressibility being considered, other factors remaining unchanged.

Two types of measuring devices will be considered. In one type, the jet is confined and transverse expansion is prevented; while in the other, the jet is free to expand both laterally and axially. A venturi tube is an example of the first, and a square-edged orifice is an example of the second.

For measurements in which a high degree of accuracy is required, certain corrections would be required, the application of which will not be considered in the material which follows. Many of the gases depart somewhat from the laws of the *ideal* gas. This phenomenon is known as supercompressibility and its effect becomes more pronounced at the higher pressures. A temperature correction would be needed to compensate for the expansion of the pipe and of the metering device. The discharge is not entirely independent of the viscosity of the gas which is to be metered. A correction would also be needed due to the fact that the temperature in the room at the location of the gages and temperature of the flowing gas

would normally differ somewhat, making the true pressure difference vary slightly from the indicated value.

75. Flow with Small Pressure Differences. — In obtaining the equations for the flow of gases through measuring devices, certain simplifying assumptions are normally made. Since the velocity of the flowing gas is high and the distance between the initial and the reduced sections is small, very little heat can be transferred as the gas passes from the first to the second point. For this reason, the flow is considered to follow the adiabatic law of expansion. It has already been shown in Arts. 64 and 65 that the velocity and discharge become independent of the viscosity for large values of Reynolds numbers. Since gas measuring devices only operate for high values of Reynolds numbers, it is reasonable to neglect the effect of viscosity. It is also assumed that the flow is steady and that the device is either horizontal or that the weight of the gas between the two sections is negligible so that $Z_1 = Z_2$.

These simplifying assumptions make possible the use of the steady flow equation for gases which appeared as Eq. (54), namely

$$\frac{V_1^2}{2g} + Z_1 = \frac{V_2^2}{2g} + Z_2 + \int_{p_1}^{p_2} v dp \tag{54}$$

By making use of the assumptions outlined above, we obtain the equivalent form

$$\frac{V_2^2 - V_1^2}{2g} = \int_{p_2}^{p_1} v dp \tag{127}$$

In order to evaluate the definite integral, it is customary to assume that the flow through the meter follows the adiabatic law, namely that

$$pv^k = \text{constant} = C \tag{128}$$

where k is the ratio of specific heat at constant pressure to that at constant volume. The value of k is about 1.13 for wet steam, 1.3 for superheated steam and natural gas and 1.4 for air. Solving for v, we obtain

$$v = C^{1/k} p^{-1/k}$$

Substituting the value of v in Eq. (127), it follows that

$$\frac{V_2^2 - V_1^2}{2g} = C^{1/k} \int_{p_2}^{p_1} p^{-1/k}\, dp$$

$$\frac{V_2^2 - V_1^2}{2g} = C^{1/k} \left(\frac{k}{k-1}\right) [p_1^{(k-1)/k} - p_2^{(k-1)/k}]$$

$$= p_1 v_1 \left(\frac{k}{k-1}\right) \left[1 - \left(\frac{p_2}{p_1}\right)^{(k-1)/k}\right] \tag{129}$$

Using the equation of continuity $w_1A_1V_1 = w_2A_2V_2$ and remembering from Eq. (128) that

$$\frac{v_2}{v_1} = \left(\frac{p_1}{p_2}\right)^{1/k}$$

the velocity and the weight discharge can be obtained. In these equations, the appropriate coefficients have been introduced.

$$V_2 = C_v\sqrt{\frac{2g(p_1v_1)\left(\dfrac{k}{k-1}\right)\left[1-\left(\dfrac{p_2}{p_1}\right)^{(k-1)/k}\right]}{1-\left(\dfrac{A_2}{A_1}\right)^2\left(\dfrac{p_2}{p_1}\right)^{2/k}}} \tag{130}$$

The weight discharge, in pounds per second, equals $w_2A_2V_2$. The conditions at the inlet can be more easily determined than those at point (2), so w_2 will be expressed in terms of w_1, or

$$w_2 = w_1\left(\frac{p_2}{p_1}\right)^{1/k}$$

These relationships can now be combined to obtain the discharge equation

$$W = C_dA_2\sqrt{\frac{2gp_1w_1\left(\dfrac{p_2}{p_1}\right)^{2/k}\left(\dfrac{k}{k-1}\right)\left[1-\left(\dfrac{p_2}{p_1}\right)^{(k-1)/k}\right]}{-\left(\dfrac{A_2}{A_1}\right)^2\left(\dfrac{p_2}{p_1}\right)^{2/k}}} \tag{131}$$

where W is the discharge in pounds per second and C_d depends upon the particular type of measuring device under consideration.

Equation (131) is cumbersome to use and it can be greatly simplified by putting it in the form of a hydraulic formula. We rewrite Eq. (131) as follows:

$$W = CA_2\sqrt{\frac{2gw_1(p_1-p_2)}{1-\left(\dfrac{A_2}{A_1}\right)^2}}$$

$$\frac{Cd}{C}\sqrt{\frac{p_1\left(\dfrac{p_2}{p_1}\right)^{2/k}\left(\dfrac{k}{k-1}\right)\left[1-\left(\dfrac{p_2}{p_1}\right)^{(k-1)/k}\right]\left[1-\left(\dfrac{A_2}{A_1}\right)^2\right]}{(p_1-p_2)\left[1-\left(\dfrac{A_2}{A_1}\right)^2\left(\dfrac{p_2}{p_1}\right)^{2/k}\right]}}$$

or

$$W = CMY\sqrt{2gw_1(p_1-p_2)} \tag{132}$$

where C is the coefficient for the same device with water flowing at high Reynolds numbers,
$$M = \frac{A_2}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

and
$$Y = \frac{Cd}{C}\sqrt{\frac{p_1\left(\frac{p_2}{p_1}\right)^{2/k}\left(\frac{k}{k-1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{(k-1)/k}\right]\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]}{(p_1 - p_2)\left[1 - \left(\frac{A_2}{A_1}\right)^2\left(\frac{p_2}{p_1}\right)^{2/k}\right]}}$$

Messrs. Bean and Buckingham of the U. S. Bureau of Standards, have done much to evaluate the factor Y for flow through venturi tubes, flow nozzles and square-edged pipe orifices. Values of Y for venturi tubes, flow nozzles and square-edged pipe orifices are given in Fig. 108[1]. Y is known as the "expansion factor" and corrects for the axial expansion which occurs in the venturi tube or in the flow nozzle, and for both the axial and transverse expansion that takes place in the unconfined jet from the orifice.

In the case of the orifice, the numerical value of the expansion factor, Y, is dependent upon the kind of taps used for obtaining the pressure. The values given in Fig. 108 may be used with corner, throat, or flange taps. They are not to be used with pipe taps.

Illustrative Problem: A 6 × 4 in. venturi tube is used to measure the rate of flow of dry air under the following conditions: $D_1 = 6.05$ in., $D_2 = 4.03$ in., $p_1 = 84.3$ lb. per sq. in. gage, $p_2 = 60$ lb. per sq. in. gage and $T_1 = 86°$ F. Find the weight discharge.

This problem will be solved first by means of the general equation, and then by use of the hydraulic type formula and the expansion factor. It will be assumed that the barometric pressure is 14.7 lb. per sq. in. and that the coefficient of discharge is 0.98.

$$W = (0.98 \times 0.0883)\sqrt{\frac{2g(99 \times 144)(0.49)\left(\frac{74.7 \times 144}{99 \times 144}\right)^{2/1.4}\left(\frac{1.4}{0.4}\right)\left[1 - \left(\frac{74.7 \times 144}{99 \times 144}\right)^{0.4/1.4}\right]}{1 - \left(\frac{4.03}{6.05}\right)^4\left(\frac{74.7 \times 144}{99 \times 144}\right)^{2/1.4}}}$$

in which $w_1 = \frac{p_1}{RT_1} = \frac{99 \times 144}{53.34 \times 546} = 0.49$ lb. per cu. ft.

$$W = 0.98 \times 0.0883\sqrt{\frac{2g(14260)(0.49)(0.668)(3.5)[1 - 0.9226]}{1 - (0.1979)(0.668)}}$$

$= 26.5$ lb. per sec. *Ans.*

[1] The values used in preparing Fig. 108 were obtained from curves which appeared in "Fluid Meters, Their Theory and Application," *A.S.M.E. Research Publication*, 4th Ed., 1937.

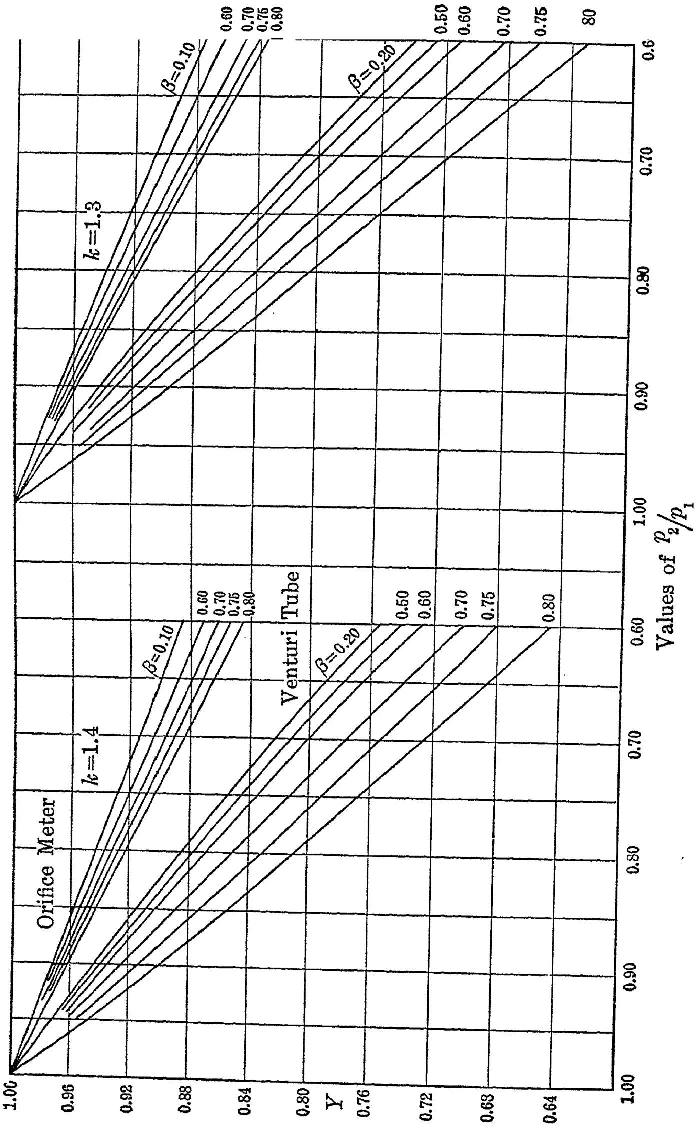


FIG. 108. Gas expansion factors.

Note that the pressures might have been expressed as pounds per square inch wherever they formed a ratio.

The same problem will now be solved by the use of Eq. (132).

$$W = 0.98(0.0986)(0.824)\sqrt{2g(0.49)(24.3)(144)}$$

$$= 26.5 \text{ lb. per sec.} \qquad \textit{Ans.}$$

where Y has been found by the use of Fig. 108, and $M = \dfrac{A_2}{\sqrt{1 - \left(\dfrac{D_2}{D_1}\right)^4}} = 0.0986.$

PROBLEMS

146. Air flows through a 12 in. $\times$ 6 in. venturi tube, for which $C_d = 0.98$, under the following conditions: $p_1 = 80$ lb. per sq. in. gage, $T_1 = 85°$ F. and $p_2 = 60$ lb. per sq. in. gage. (Use $k = 1.4$ and $R = 53.34$.) Find the weight discharge using Eq. (131).

147. Solve Prob. 146 using Eq. (132).

148. Air flows through an 8 in. $\times$ 4 in. venturi tube, having a coefficient of discharge of 0.976, with a manometer reading of 30 in. of water. $p_1 = 50$ lb. per sq. in. gage and $T_1 = 75°$ F.

Find the weight discharge (*a*) neglecting the effect of compressibility, and (*b*) considering the effect of compressibility.

Ans. (*a*) $W = 5.04$ lb. per sec., (*b*) $W = 4.99$ lb. per sec.

149. A 4 in. square-edged orifice is installed in a 12 in. pipeline. Superheated steam for which $p_1 = 200$ lb. per sq. in. abs., $T_1 = 440°$ F. and $v_1 = 2.510$ cu. ft. per lb. flows through the orifice with a pressure drop of 15 lb. per sq. in. Find the weight discharge.

150. Find the value of the throat velocity in Prob. 146.

151. Find the pressure drop across a 6 in. orifice installed in a 12 in. line which carries air at the rate of 40 lb. per sec. $p_1 = 100$ lb. per sq. in. abs., $T_1 = 70°$ F. and the water coefficient of the meter is 0.62.

152. A 3 in. flow nozzle is attached to the end of a 12 in. pipe which carries air at a pressure of 5 lb. per sq. in. gage and a temperature of 90° F. Assuming no contraction of the jet and a reasonable coefficient, find the weight discharge.

Ans. $W = 2.75$ lb. per sec.

153. Find the jet velocity for the condition outlined in Prob. 152.

76. Flow with Large Pressure Differences. — Inspection of Eq. (131) shows that the weight discharge will be zero when $p_2 = p_1$ and also when $p_2 = 0$. Since this is the case, there will be some intermediate pressure ratio for which the discharge will be a maximum. Should we consider the discharge from a large tank so that the velocity of approach correction could be omitted, the maximum value of the weight discharge could be determined by finding the maximum value of

$$\left(\frac{p_2}{p_1}\right)^{2/k}\left[1 - \left(\frac{p_2}{p_1}\right)^{(k-1)/k}\right]$$

in which p_2 will be considered the variable. By differentiating and equat-

ing the first derivative to zero, we find that the critical value of the pressure p_2 (namely $p_{cr.}$) is given by the expression

$$p_{cr.} = p_1\left(\frac{2}{k+1}\right)^{k/(k-1)} \tag{133}$$

If this value of the critical pressure is now used in finding the velocity of efflux, V_2, from the large tank (velocity of approach correction omitted), we find

$$V_2 = \sqrt{gkp_{cr.}v_{cr.}} \tag{134}$$

Equation (134) is also that of the velocity of sound in the same medium, that is, in a medium having the same values for k, p, and v. For the solution of V_2, it is more convenient to use the conditions at the inlet. Substituting $p_{cr.}$ in Eq. (130) with the velocity of approach correction omitted, it follows that

$$V_2 = C_v\sqrt{2g\left(\frac{k}{k+1}\right)p_1v_1} \tag{135}$$

The weight discharge will be obtained by the relationships given in equations (133) and (134).

$$W = C_dA_2\sqrt{2gw_1p_1\left(\frac{k}{k+1}\right)\left(\frac{2}{k+1}\right)^{2/(k-1)}} \tag{136}$$

Let us now investigate the logic which is expressed by the last three equations. By Eq. (133), we found that the weight discharge from a device was a maximum at any one section when the pressure at that section reached the critical value. Let us assume that the flow is through a section which is surrounded by a solid boundary, such as would be the case for a venturi tube or a nozzle. Now let us assume that the pressure at the throat is to drop below the critical value. Should this be the case, the pressure at some section upstream from the throat would be equal to the critical value and the discharge at this larger section would be a maximum. Since the equation of continuity must be satisfied, a greater discharge would have to flow through the throat than could exist according to Eq. (136); a condition which would be manifestly impossible.

Equation (136) applies to a venturi tube or nozzle in which the flow is confined from the inlet to the throat. It does not apply in the case of the square-edged orifice in which the jet is not confined.

Hartshorn[1] measured the discharge from a thin lipped orifice and concluded that the weight discharge continued to increase until the pressure ratio decreased to 0.2. The method which Hartshorn used in measuring

[1] Hartshorn, L., "The Discharge of Gases under High Pressures," *Proc. Royal Society A.*, V. 94, pp. 155–165, 1918.

the discharge did not result in a high degree of accuracy, so later and more accurate experiments were conducted by Stanton[1] on a square-edged orifice having an area ratio of 0.0466 and he found a steady increase in discharge as the pressure ratio decreased to about 0.24. Stanton investigated the conditions within the jet by means of a Pitot tube and concluded that the air was flowing with the velocity of sound, but that the increase in weight discharge was due to a measurable increase in the cross-sectional area of the jet. More recently, Schiller[2] conducted a more complete series of tests on orifices having area ratios of 0.144, 0.337, 0.423 and 0.577, and found an increase in discharge comparable to that found by Stanton. Schiller concludes that the discharge would increase until p_2 were zero.

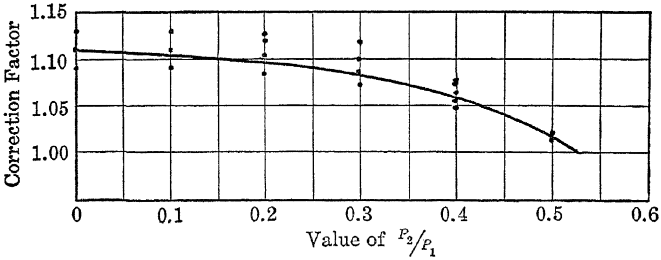


FIG. 109. Correction factor for gas flow through orifices for large pressure differences.

The correction curve, Fig. 109, has been prepared from the curves presented by Stanton and Schiller for the flow of air through square-edged orifices. This curve gives a correction coefficient by which the computed weight discharge for $p_2 = p_{cr.}$ would be multiplied in order to obtain the actual discharge.

For values of $p_2 \leqq p_{cr.}$, the weight discharge for a device such as a venturi tube, nozzle or well rounded orifice would be found by the direct application of Eq. (136). The discharge of air from a square-edged orifice would be obtained by first finding the value given by Eq. (136) and then correcting by the appropriate coefficient which would be obtained from Fig. 109. While Fig. 109 has been obtained from tests on the discharge of air, it is probable that the same curve could be used for other gases without the introduction of serious error.

While it has been shown that the maximum velocity of the gas in the measuring devices considered above was equal to the velocity of sound in the same medium, it is possible for velocities to exceed the velocity of sound.

[1] Stanton, T. E., "On Flow of Gases at High Speeds," *Proc. Royal Society A.*, V. 111, pp. 306–339, 1926.

[2] Schiller, W., "Überkritische Entspannung kompressibler Flussigkeiten," *Forschung auf dem Gebiete des Ingenieurwessens*, V. 4, pp. 128–137, 1933.

Such a condition may exist in a nozzle which first contracts and then expands as the nozzles of steam turbines, may exist in the flow through the valves of an engine, and does exist in the discharge of explosives. These high velocities are known as supersonic velocities and it is felt that a consideration of them is beyond the scope of this text.

PROBLEMS

154. Air flows from a large tank, in which $p_1 = 120$ lb. per sq. in. abs. and $T_1 = 80°$ F., through a well rounded orifice, which is 1.2 in. in diameter, out into the atmosphere.

Find the critical pressure for this condition and the velocity of efflux.

155. Find the weight discharge for the conditions outlined in Prob. 154.

156. A 3 in. square-edged orifice is placed in a 12 in. pipeline in which air is flowing. $p_1 = 80$ lb. per sq. in. abs., $T_1 = 70°$ F., $p_2 = 25$ lb. per sq. in. abs.

Find the weight discharge.

Ans. $W = 8.59$ lb. per sec.

CHAPTER VIII

PIPE FLOW

77. Introduction. — It is hardly necessary to remind the reader of the importance of pipes in our modern civilization. We are dependent upon them for conveying water, oil, gas, and numerous other fluids so necessary for our domestic comfort and industrial well being. There are very few projects of an engineering nature which do not require a knowledge of pipe flow.

A pipeline is usually made up of sections of pipe of constant or differing diameter joined by various types of fittings manufactured commercially for this purpose. Regulation of the flow is accomplished ordinarily by valves placed at strategic points along the pipeline. Various of the more common types of fittings and valves are shown in Fig. 110.

A pipeline is distinguished from an open channel by the fact that all the fluid within the pipeline is under a pressure usually differing from atmospheric; whereas in open channel flow, the surface of the liquid is subjected to the pressure of the atmosphere.

In Chap. V, it was stated that the flow of real fluids cannot take place without a loss of energy. It is the purpose of this chapter to study the nature and amount of the energy losses occurring when fluids flow through pipes. The first part of the discussion will be limited to the flow of liquids, and to the flow of gases with relatively small pressure changes. Under these conditions, gases may be treated as non-compressible without serious error.

78. Laminar and Turbulent Flow. — That fluids may flow in two distinct ways was first demonstrated by Osborne Reynolds.[1] An apparatus similar to the one used by him is shown in Fig. 111. It consists of a bell-mouthed glass tube inserted in a supply tank having glass sides. The tank contains water which has been allowed to become quiescent. At the bell-mouth end of the glass tube, means for injecting a fine thread of dye are provided. Flow through the tube is regulated by a valve near the discharge end.

If the control valve is opened gradually, the filament of dye will remain sharply defined and perfectly straight up to a certain average velocity which is found to depend upon the degree of quiescence of the water in the

[1] Reynolds, Osborne, "An Experimental Investigation of the Circumstances which Determine whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," *Phil. Trans. Royal Soc.*, V. 174, Part III, 1883.

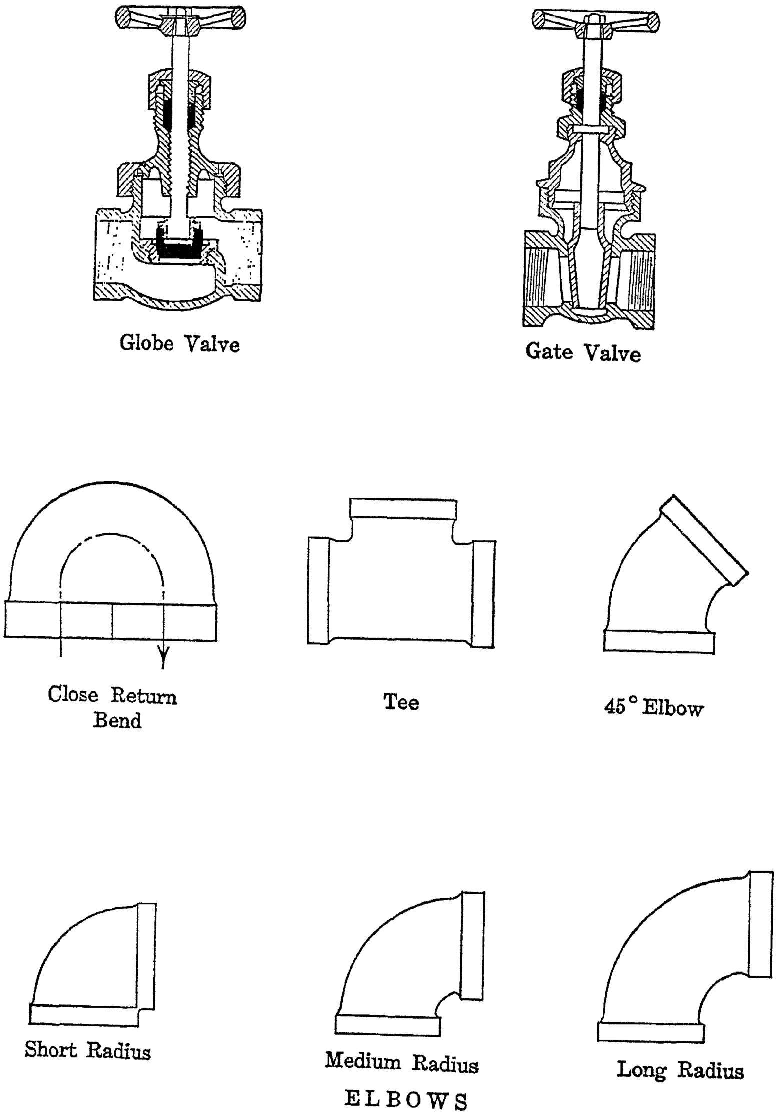


FIG. 110. Types of valves and pipe fittings.

supply tank and the existence or absence of local disturbances within the tube. In this type of motion, the particles of fluid move in well defined rectilinear paths, and there is no intermingling of the particles. This type of flow has been previously defined as *laminar flow*.

When the velocity of flow is further increased, the filament of dye begins to waver, and with a still further increase, breaks down at some distance from the bell mouth with the dye diffusing throughout the liquid in an unorderly way. When this happens, the motion of the individual particles is very irregular and the tracing of the path of the individual particles is

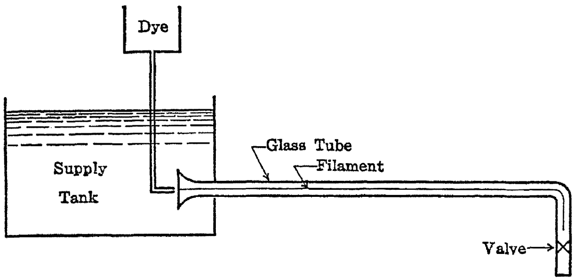


FIG. 111. Reynolds' apparatus.

very difficult. In addition to the component of velocity of a particle in the general direction of flow, the velocity will have a cross-current component at any instant. This is called *turbulent motion*.

There seems to be some question as to the point in the cross section at which turbulence first appears. Experiments by Gibson[1] indicate that turbulence is initiated at a distance equal to about 0.6 of the radius from the center. Turbulence once begun, however, spreads quickly throughout the cross section with very little increase in velocity. The average velocity at which the flow changes from laminar to turbulent is called the *upper critical velocity*. This velocity is not well defined, depending upon the initial condition of quiescence and the existence or lack of local disturbances.

If the process described above is reversed, and the flow conditions are changed from those of turbulence to those of laminar flow by reducing the velocity, it will be found that, regardless of any small disturbances, the flow will revert to the laminar type at a certain definite velocity. Below this velocity, even externally produced turbulence will be damped out very quickly. The velocity at which this occurs is comparatively well defined, and is called the *lower critical velocity*.

[1] Gibson, A. H., "The Breakdown of Streamline Motion at the Higher Critical Velocity in Pipes of Circular Cross Section," *Phil. Mag. and Jl. of Sc.*, V. 15, 7th Series, p. 637, 1933.

The method described above for showing the existence of the two types of flow, while convincing and spectacular, leads to no information regarding the energy losses accompanying each type of flow. A more productive experiment for this purpose may be performed in the following manner. Figure 112 shows the apparatus required. It consists of a pipe of known

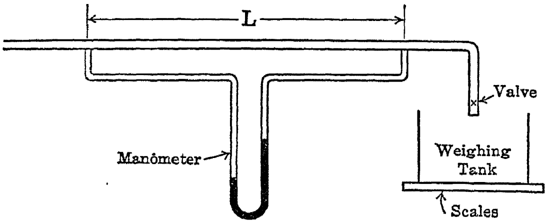


FIG. 112. Set up for measuring pipe friction.

uniform diameter to which a manometer is connected for measuring the head loss occurring in a length L. The sample length is situated far enough down stream from any valve or fitting so that the flow can assume steady conditions. A valve near the outlet of the pipe regulates the flow, and the discharge occurring during a measured interval of time is weighed by means of a tank and scale.

At any particular discharge, the head loss is computed from the manometer reading, and the average velocity is determined from the measured weight discharged. At other discharges, the velocity and head loss may be determined in the manner described above. In this way, sufficient data are obtained for plotting a curve of head loss, h_f, against the velocity V. For reasons to be explained, it will be found better to plot the logarithms of the variables h_f and V rather than the values themselves. Double logarithmic paper is suitable for this purpose. A curve of this type is shown in Fig. 113. The curve consists of two parts which plot approximately as straight lines, separated by a group of points which have no well defined trend. The portion AB, for the lower velocities, has a slope of 1; and DE, a slope of about 1.8 or more, but less than 2. The portion AB corresponds to laminar flow and DE to turbulent flow.

Assuming, as is often done, that the head loss can be expressed by the equation

$$h_f = kV^n \tag{137}$$

in which k is a constant for the given pipe diameter and length, and writing this equation in logarithmic form, we obtain

$$\log h_f = n \log V + \log k \tag{138}$$

If we substitute y for $\log h_f$, x for $\log V$, and c for $\log k$, Eq. (138) may be

written

$$y = nx + c \tag{139}$$

Equation (139) is the equation of a straight line with a slope n. This explanation shows that if the plot of log h_f and log V results in a straight line, Eq. (137) is a correct expression for head loss and the exponent n is the slope of the line.

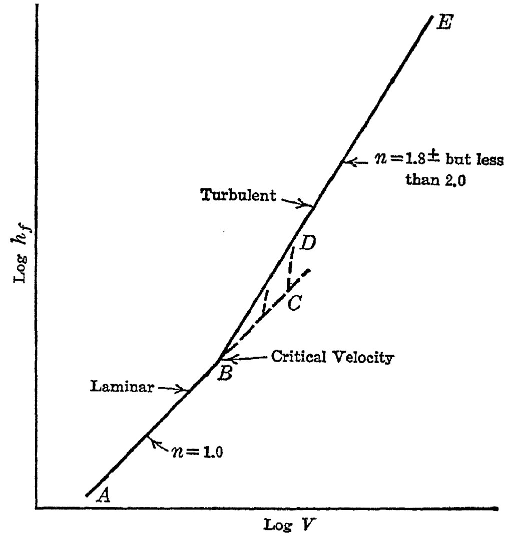


FIG. 113. Typical conditions at the critical velocity for pipe flow.

Since the value of n in Fig. 113 is 1 for laminar flow, the head loss for this type of flow is proportional to the first power of the velocity. For turbulent flow, the head loss is proportional to slightly less than the second power of V, depending upon the roughness of the pipe.

Although the method used in determining n described above is experimental, the same conclusion, at least with respect to laminar flow may be reached by purely theoretical reasoning, as will be shown later.

79. Criterion for Determining Type of Flow. — Osborne Reynolds, from both theoretical and experimental reasoning, deduced that the true criterion for determining the type of flow to be expected in a circular pipe did not merely depend upon the value of the velocity but rather upon the value of the dimensionless number

$$\mathbf{R} = \frac{DV\rho}{\mu}$$

in which $\mathbf{R}$ = Reynolds number, dimensionless,
D = diameter, feet,
V = average velocity, feet per second,
ρ = density, slugs per cubic foot,
μ = absolute viscosity, pounds second per square foot.

Any system of units for the dimensions will give the same Reynolds number provided that they are consistent among themselves.

For flow taking place with $\mathbf{R}$ less than 2000, the flow is generally laminar. With a Reynolds number larger than 2000, the flow in commercial pipes under practical conditions should be assumed to be turbulent. Under certain ideal conditions, laminar flow can be maintained with Reynolds number well above 30,000, but any small disturbance will disrupt the flow if the velocity is above the lower critical value. In practice, such disturbances are always assumed to exist.

Reynolds number enables us to determine the type of flow in a pipe of any diameter and for any kind of fluid. Thus, the variables upon which the type of flow generally depends are grouped in a single ratio. The recognition of the relationship represented by Reynolds number constituted an important step forward in the study of fluid flow.

It has been stated that, in most cases of practical importance, the flow is turbulent. That this is so, especially when water is flowing, may be shown by the following computation. Lea[1] gives 3 ft. per sec. as approximately the average velocity in a pipeline carrying water. Assuming a 1 in. diameter pipe and a temperature of 60° F., $\rho = 1.94$ slugs per cu. ft. and $\mu = 2.36 \times 10^{-5}$ lb. sec. per sq. ft.

$$\mathbf{R} = \frac{DV\rho}{\mu} = \frac{1 \times 3 \times 1.94}{12 \times 2.36 \times 10^{-5}} = 20{,}500$$

Since this value exceeds the critical Reynolds number, the flow is turbulent. In dealing with fluids of relatively high viscosity, such as heavy oils, laminar flow may be expected even in pipes of relatively large diameter.

PROBLEMS

157. Oil with a specific gravity of 0.92 and an absolute viscosity of 6×10^{-3} lb. sec. per sq. ft. is flowing in a 2 in. diameter pipe at the rate of 0.5 c.f.s. (*a*) What is the value of $\mathbf{R}$? (*b*) Is the flow turbulent or laminar?

158. Water at a temperature of 68° F. flows through a ½ in. copper tube (actual inside diameter 0.378 in.). What weight flow can be permitted in 4 minutes without causing turbulent flow to exist in the pipe? *Ans.* $W = 8.15$ lb.

[1] Lea, F. C., "Hydraulics," 5th Ed., p. 240, Longmans, Green and Co.

80. Forces Involved in Pipe Flow. — The free body sketch of a portion of fluid included between two cross sections of a pipe a distance L apart is illustrated in Fig. 114. The pipe is taken in a horizontal position for simplicity. Since the flow is assumed steady and the pipe is of uniform diameter, the velocity heads, and also the elevation heads, at (1) and (2) are equal. Therefore, any loss of energy will make itself known by a reduction of pressure head so that the pressure at (2) will be less than that at (1). The average pressures at (1) and (2) are indicated by p_1 and p_2. Since the mass being considered is in steady motion, the resultant of the forces acting must be zero. This requires that we have a shearing force at the wall indicated by τ_o in lb. per sq. ft. The radius of the pipe is r_o. Using the equation $\sum F_x = 0$, we have

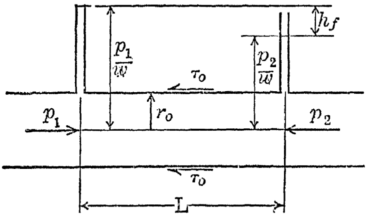


FIG. 114

$$(p_1 - p_2)\pi r_o^2 = 2\pi r_o L \tau_o$$

and

$$p_1 - p_2 = \frac{2L\tau_o}{r_o} \tag{140}$$

The frictional head loss, in feet of the fluid flowing, is given by

$$h_f = \frac{p_1 - p_2}{w} = \frac{2L\tau_o}{wr_o} \tag{141}$$

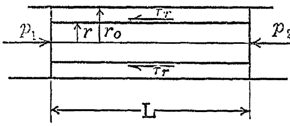


FIG. 115.

For a cylindrical portion of fluid of radius r (Fig. 115), a similar analysis will give

$$h_f = \frac{p_1 - p_2}{w} = \frac{2L\tau_r}{wr} \tag{142}$$

in which τ is the unit shearing stress at a distance r from the center of the pipe. Equation (142) indicates that the loss in energy can be determined quite readily if the shearing stress can be determined at any point in the fluid. From equations (141) and (142), it may be shown that the shearing stress varies directly as the distance from the center, or

$$\frac{\tau_r}{\tau_o} = \frac{r}{r_o} \tag{143}$$

PROBLEMS

159. Water flows in a 4 in. horizontal pipe. The pressure drop in 100 ft. of pipe is 2.0 lb. per sq. in. (*a*) What is the shearing stress at the wall of the pipe? (*b*) What is the shearing stress 1 in. from the center?

Ans. (*a*) $\tau_o = 0.24$ lb. per sq. ft., (*b*) $\tau_r = 0.12$ lb. per sq. ft.

160. Oil with a specific gravity of 0.85 flows in a 6 in. pipe; the shearing stress at the wall is known to be 0.161 lb. per sq. ft. What is the head loss in 200 ft. of pipe?

81. Darcy or Weisbach Equation for Head Loss in Pipes. — In the last article, it was shown that the head loss in steady uniform flow could be computed if the distribution of shearing stress in the pipe could be determined. This is possible analytically only in the case of laminar flow, as will be demonstrated in Art. 82. For the more important and complicated condition of turbulent flow, experimental methods are necessary. In either case, a dimensional analysis, if based on correct assumptions, should divulge the general form of the equation for shearing stress.

Assuming that the shearing stress at the wall, τ_o, is dependent upon the density ρ, viscosity μ, velocity V, and the diameter of the pipe D, the dimensional analysis of Art. 52 gave the following equation

$$\tau_o = \rho V^2 \phi(\mathbf{R}) \tag{62}$$

The character and size of wall roughness has been ignored in deriving Eq. (62) so that the equation is correct only if the effect of roughness is negligible or if the wall of the pipe is perfectly smooth. The kind and size of wall roughness are by no means insignificant in most cases, but we shall leave discussion of these to a later article.

Substituting the value of τ_o given by Eq. (62) in Eq. (141) for the head loss, there results

$$h_f = \frac{2L}{wr_o}\rho V^2 \phi(\mathbf{R})$$

Since $D = 2r_o$ and $\rho/w = 1/g$, the above equation for head loss may be written as

$$h_f = 8\phi(\mathbf{R})\frac{L}{D}\frac{V^2}{2g} = \phi'(\mathbf{R})\frac{L}{D}\frac{V^2}{2g}$$

Let

$$\phi'(\mathbf{R}) = f$$

then

$$h_f = f\frac{L}{D}\frac{V^2}{2g} \tag{144}$$

in which f is a dimensionless number and is a function of **R**. Equation (144) is the Weisbach or Darcy equation for head loss in pipes. It is expressed in terms of the velocity head in order that it may be more readily used in the Bernoulli equation. Equation (144) is perfectly general and applies to either turbulent or laminar flow provided only that the assumptions made in the derivation are satisfied.

The form of the function

$$f = \phi'(\mathbf{R})$$

cannot ordinarily be expressed algebraically, and is most often obtained as an experimental curve. Thus, for a given pipe of diameter D and length L, the head loss and discharge may be measured. f can then be computed from Eq. (144) and **R** determined. f is then plotted against **R** to locate one point on the curve $f = \phi'(\mathbf{R})$. By varying the discharge, other values for the curve may be found in a similar manner.

PROBLEMS

161. In an experiment for determining $f = \phi'(\mathbf{R})$ for a $\frac{1}{2}$ in. galvanized pipe discharging water, the following data were obtained:

Length between manometer connections	18.5	ft.
Actual diameter of pipe	0.602	in.
Discharge in 2 minutes	15.00	lb.
Differential manometer deflection	8.00	in.
Manometer fluid CCl_4 with S.G. =	1.60	

What were the values of f and **R**?

162. In Art. 82 it will be shown that $f = 64/\mathbf{R}$ for laminar flow. Using this value of f, find the maximum head loss permissible in 17 ft. of $\frac{1}{2}$ in. copper tube, inside diameter = 0.38 in., without causing turbulent flow. The fluid is oil with a kinematic viscosity of 0.005 sq. ft. per sec.

82. Head Loss in Laminar Flow. — Laminar flow does not occur in practice except when highly viscous fluids such as heavy oils and molasses are involved. However, a knowledge of the law of energy loss in this type of flow is important, and its study leads to a better conception of the whole problem of fluid flow.

We proceed to derive an expression for the head loss in laminar flow, and to show that it is in the form

$$h_f = \phi(\mathbf{R}) \frac{L}{D} \frac{V^2}{2g}$$

Figure 116 represents laminar flow of a fluid in a circular pipe of radius r_o. Taking a cylindrical portion of the fluid of radius r and length L, and

writing the equilibrium equation in the direction of flow, we have

$$\pi r^2(p_1 - p_2) = \tau_r 2\pi r L$$

or

$$\tau_r = \frac{(p_1 - p_2)r}{2L} \tag{a}$$

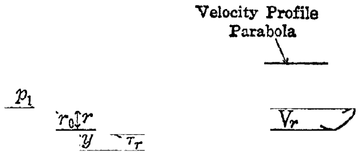


FIG. 116. Laminar flow in circular pipe.

τ_r is the unit shearing stress at a distance r from the center of the pipe. From the definition of viscosity,

$$\tau_r = \mu \frac{dV}{dy}$$

Since

$$r = r_o - y, \quad dy = -dr \quad \text{and} \quad \tau_r = -\mu \frac{dV}{dr}$$

Equation (a) becomes

$$-\mu \frac{dV}{dr} - \frac{(p_1 - p_2)r}{2L}$$

$$-dV = \frac{(p_1 - p_2)}{2\mu L} r dr \tag{b}$$

Equation (b) is the differential equation for the velocity profile. Integrating

$$-V\Big]_{V_{\text{wall}}}^{V_r} = \frac{(p_1 - p_2)r^2}{4\mu L}\Big]_{r_o}^{r} \tag{c}$$

All of the experimental evidence indicates that the velocity of the fluid directly at the wall is zero. Assuming this, and calling the velocity at radius r, V_r, Eq. (c) gives

$$V_r = \frac{(p_1 - p_2)}{4\mu L}(r_o^2 - r^2) \tag{d}$$

The velocity at the center, V_c, is given by substituting $r = 0$ in Eq. (d).

$$V_c = \frac{(p_1 - p_2)}{4\mu L} r_o^2$$

or comparing with Eq. (d).

$$V_r = V_c - \frac{(p_1 - p_2)}{4\mu L} r^2 \tag{e}$$

Equation (e) shows that the velocity distribution follows the parabolic law and that a plot of the velocities would form a paraboloid of revolution. Since the discharge is proportional to the volume of the velocity figure and since the volume of a paraboloid is one-half the volume of the circumscribed cylinder, the discharge is

$$Q = \tfrac{1}{2}\pi r_o^2 \, V_c = \pi r_o^2 \, V_{\text{ave.}}$$

from which the average velocity is one-half the center velocity.

$$V_{\text{ave.}} = \frac{1}{2}\frac{(p_1 - p_2)}{4\mu L} r_o^2 = \frac{(p_1 - p_2)}{8\mu L} r_o^2$$

Indicating the average velocity by V,

$$p_1 - p_2 = \frac{8\mu L V}{r_o^2}$$

and

$$h_f = \frac{p_1 - p_2}{w} = \frac{8\mu L V}{r_o^2 w} \tag{f}$$

Since $w = \rho g$ and $r_o = D/2$, Eq. (f) can be written

$$h_f = \frac{8\mu L V^2 2}{\frac{D^2}{4}\rho g V 2} = \left(\frac{64\mu}{VD\rho}\right)\frac{L}{D}\frac{V^2}{2g}$$

or

$$h_f = \left(\frac{64}{\mathbf{R}}\right)\frac{L}{D}\frac{V^2}{2g} \tag{145}$$

Equation (145) is the equation for head loss for laminar flow through a circular pipe. Since $\mathbf{R}$ is $DV\rho/\mu$, the equation shows that the energy loss is proportional to the first power of the velocity, as was previously stated. Also, the equation is in the form

$$h_f = \phi(\mathbf{R})\,\frac{L}{D}\frac{V^2}{2g}$$

in which $\phi(\mathbf{R}) = f = \dfrac{64}{\mathbf{R}}$.

PROBLEMS

163. Oil with a specific gravity of 0.9 and absolute viscosity of 3×10^{-3} lb. sec. per sq. ft. is flowing through a 2 in. pipe at the rate of 0.3 c.f.s. (*a*) Find the head loss in 200 ft. of pipe. (*b*) Find the shearing stress at the wall.

164. A mercury-oil manometer is connected to a 1 in. pipe at points 100 ft. apart. The manometer shows a deflection of 1 in. when the discharge through the pipe is 20 lb. per min. The fluid is an oil having a specific gravity of 0.85. What is the absolute viscosity of the oil?

165. Derive an expression for the head loss when a fluid flows laminarly between two parallel plates a distance h apart in terms of the average velocity.

$$h_f = \frac{p_1 - p_2}{w} = \frac{12\mu LV}{h^2 w} \quad Ans.$$

166. An oil having a specific gravity of 0.90 and an absolute viscosity of 2×10^{-4} lb. sec. per sq. ft. flows laminarly between two parallel plates 1 in. apart. What is the head loss in a distance of 20 ft. parallel to the direction of flow when the velocity is 0.5 ft. per sec? See Prob. 165.

83. Velocity Distribution in Turbulent Flow. — The study of laminar flow is comparatively simple because the individual particles of the fluid travel in rectilinear paths. At any instant, the velocity is in the direction of flow and there is no transverse component. Under these conditions, and with the definition of viscosity given, it was a simple matter to show that the velocity profile was parabolic. Knowing the velocity distribution, the expression for the friction factor was easily deduced.

In turbulent flow, on the other hand, the flow is steady at any point in the fluid only in the average sense. Thus at any point in the fluid, there are continual fluctuations both in a transverse and a longitudinal direction. The fluctuations have an average value of zero. Although this is true, the fluctuations have an important effect in greatly increasing the shearing stress at any point and in greatly increasing the energy losses. An additional effect of the mixing action in turbulent motion is to cause the velocity to be much more nearly uniform in the central portion of the pipe. The average velocity in turbulent flow varies usually between 0.8 and 0.85 of the maximum velocity. This is in contrast to laminar flow where the ratio is 0.5. Since the Reynolds number gives a measure of the turbulence, it is reasonable to conclude that the larger ratios of average velocity to maximum velocity in the same pipe occur with the larger Reynolds numbers. Figure 117 and Fig. 118 show the variation of velocity in a smooth pipe as given by Rouse.[1] Figure 117 shows how the velocity profile gradually becomes more uniform in the central portion as Reynolds number is increased, the average velocity remaining the same. Figure 118

[1] Rouse, Hunter, "Modern Conceptions of Fluid Turbulence." *Trans. A.S.C.E.*, V. 102, p. 463, 1937.

gives the ratio of the average velocity, V, to the maximum velocity at the center, V_c, as Reynolds number increases. The corresponding curves for other roughnesses differ somewhat from those for smooth pipes.

Since a transverse fluctuation is necessary for turbulence and since no such fluctuation can exist at the wall of a pipe, even in turbulent flow, a very thin laminar layer must exist next to the wall. Experiments in the measurements of velocities indicate that the fluid in contact with the wall has zero velocity. This assumption is usually adopted.

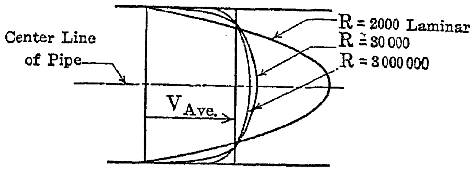


FIG. 117. Velocity distribution in smooth pipe.

The fluctuations of velocity characterizing turbulent flow make the theoretical analysis of this type of flow very difficult. The fluctuations are haphazard and depend upon factors such as the magnitude and shape of wall roughness and the value of Reynolds number. The problem has been successfully attacked in connection with flow in smooth and very rough

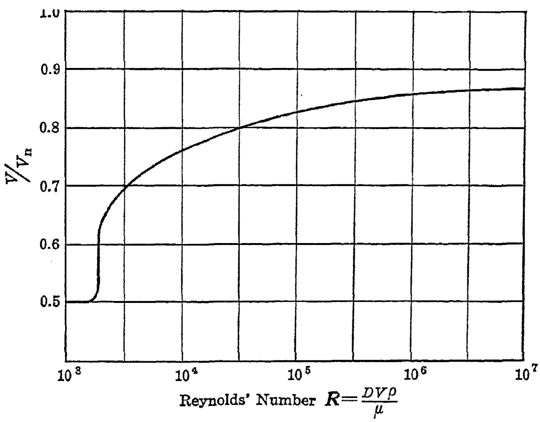


FIG. 118. Ratio of mean to maximum velocity in smooth pipe.

pipes. The work of Prandtl, Nikuradse and von Karman has led to theoretical equations for the velocity profile and shearing stress for smooth and for very rough pipes. The pipes used to check theoretical results were artificially roughened so that similarity between pipe roughnesses existed and the relative roughness could be measured. In commercial pipes, however, the roughnesses encountered have so many variations in size and shape that the determination of the friction factor for them is still experimental.

84. Head Loss in Turbulent Flow. — As stated in Art. 83, the determination of the friction factor in turbulent flow is still experimental. Until recently, investigators worked more or less independently and used water as the flowing medium. The usual procedure was to measure the head loss and discharge through a given pipe, and compute f from the equation

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

The results took the form of a table or chart in which f was given for the particular velocity in the pipe tested. Results of this kind were limited in scope and applied strictly only for water at the temperature of the experiment and in the pipe used in conducting the test. No answer was given concerning what could be expected if another fluid were discharged through the same pipe. The use of dimensional analysis, however, has brought about the generalization of results at least with regard to flow of different type fluids in the same pipe. To explain this, let us recall the equation for head loss derived under the assumption that the head loss is dependent upon viscosity, density, velocity and diameter of pipe: i.e.,

$$h_f = \phi(\mathbf{R})\frac{L}{D}\frac{V^2}{2g} = f\frac{L}{D}\frac{V^2}{2g}$$

A closer examination as to the real meaning of this equation is very important. The equation states that the friction factor f is a function of Reynolds number. It also states that the head loss in the same pipe, in feet of the fluid flowing, is the same regardless of the fluid flowing for a constant Reynolds number. Since Reynolds number is

$$\mathbf{R} = \frac{DV\rho}{\mu}$$

we may have an entirely different fluid flowing in the pipe and still retain the same Reynolds number. For this number, the friction factor to be used in the Darcy equation is the same regardless of the type of fluid. If data are obtained for plotting a curve of f against Reynolds number for a certain pipe using water as the fluid, this curve should be applicable for any fluid flowing through the same pipe. As yet, nothing has been said as to what might happen if the fluid were flowing through another pipe of different wall surface or different diameter.

Nikuradse went one step further in generalizing the relationship between f and **R**. By artificially roughening pipes with grains of sand of known diameter, he obtained geometric similarity in a series of pipes. He then proceeded to show that the ratio of the diameter of the pipe, D, to the average diameter of sand particle, ϵ, could be used as a measure of rough-

ness. As long as this ratio was the same, no matter what the diameter of pipe, the f vs. $\mathbf{R}$ curve plotted as one line. In this way, he was able to plot a curve for each roughness ratio D/ϵ. The conclusion to be drawn from the experiments of Nikuradse is that, for pipes which are geometrically similar, one curve of f vs. $\mathbf{R}$ will serve for all such pipes.

An attempt to classify commercial pipes according to geometric similarity would be practically impossible. While the ratio of average surface projection to diameter might be the same for a series of pipes, the shape and spacing of the projections would probably be entirely different, and similarity would be lacking. However, it has been found that certain types of pipe surfaces for corresponding diameters have similar characteristics in that their f vs. $\mathbf{R}$ relationship is the same. Such pipes are said to be hydraulically similar even though they may not be geometrically similar. Pigott and Kemler,[1] after analyzing experimental data for some four thousand determinations of friction factor, have classified commercial pipes hydraulically. Their results are shown in Fig. 119 and the accompanying table.

The curves in Fig. 119 are numbered from 1 to 18 in order of increasing roughness. The table groups pipes of different materials into their roughness classification. For example, all the pipes in column 4 of the table are hydraulically similar. This is true even though they may be made of entirely different materials, as shown by the classification below the table.

The use of Fig. 119 is recommended for determining friction factors in commercial pipe work. The diameters given in the table are the nominal diameters which, for small pipes, differ considerably from the actual diameters. Although a portion of the friction factor curve for laminar flow is given in Fig. 119, it is usually better to determine f for this type of flow from the relationship

$$f = \frac{64}{\mathbf{R}}.$$

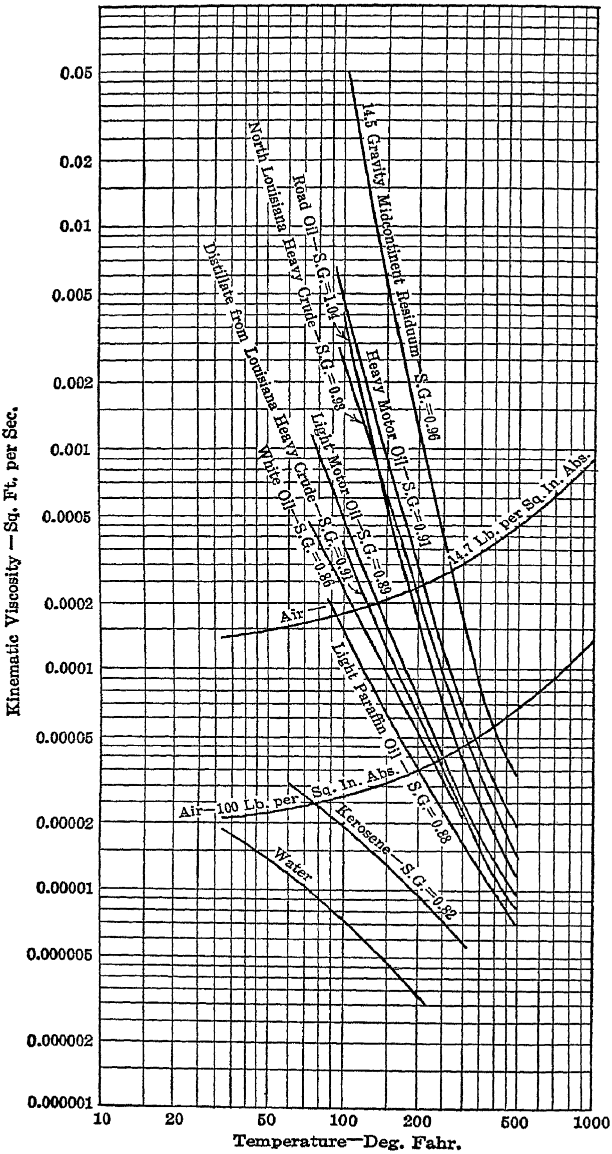
PROBLEMS

167. Water at 70° F. is flowing through a 12 in. average cast iron pipe at the rate of 2 c.f.s. What is the value of the friction factor?

168. Air at a temperature of 60° F. and pressure of 40 lb. per sq. in. abs. is flowing through a 6 in. clean steel pipe at the rate of 2 lb. per sec. Find the friction factor. R for air is 53.3. *Ans.* $f = 0.0164$.

169. Oil with a specific gravity of 0.95 and absolute viscosity of 2×10^{-3} lb. sec. per sq. ft. flows through a 3 in. galvanized pipe at the rate of 0.5 c.f.s. What is the friction factor?

[1] Kemler, Emory, "A Study of the Data on the Flow of Fluids in Pipes." *A.S.M.E. Transactions*. Vol. 55, 1933. HYD-55-2. Pigott, P. J. S., "The Flow of Fluids in Closed Conduits." *Mechanical Engineering*, Vol. 55, p. 497, 1933.



0.05
0.02
0.01
0.005
0.002
0.001
0.0005
0.0002
0.0001
0.00005
0.00002
0.00001
0.000005
0.000002
0.000001
10
20
50
100
200
500
1000
Kinematic Viscosity — Sq. Ft. per Sec.
Temperature—Deg. Fahr.
14.5 Gravity Midcontinent Residuum—S.G.=0.96
North Louisiana Heavy Crude—S.G.=0.98
Road Oil—S.G.=1.04
Distillate from Louisiana Heavy Crude—S.G.=0.91
Heavy Motor Oil—S.G.=0.91
Light Motor Oil—S.G.=0.89
White Oil—S.G.=0.86
Light Paraffin Oil—S.G.=0.88
Kerosene—S.G.=0.82
Water
Air—14.7 Lb. per Sq. In. Abs.
Air—100 Lb. per Sq. In. Abs.

FIG. 47

ROUGHNESS CLASSIFICATION

Actual inside diameter given for tubing; in pipe, nominal size of standard weight.

Curve	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	0.35 up																	
B	72	48-66	14-42	6-12	4-5	2-3	1½	1-1¼	¾	½	⅜	¼	⅛	0.125				0.0625
C			30	10-24	6-8	3-5	2½	1½-2	1¼	1	¾	½						
D			48-96	20-48	12-16	5-10	3-4	2-2½	1½	1¼	1			⅜		¼		⅛
E			96	42-96	24-36	10-20	6-8	4-5	3									
F			220	84-204	48-72	20-42	16-18	10-14	8	5	4	3						

A=drawn tubing, brass, lead, tin, glass, dia., ins. *B*=clean steel, wrought iron, dia., ins. *C*=clean galvanized, dia., ins. *D*=best cast iron, cement, light-riveted sheet ducts, dia., ins. *E*=Average cast iron, rough formed concrete, dia., ins. *F*=first class brick, heavy-riveted, spiral riveted, dia., ins.

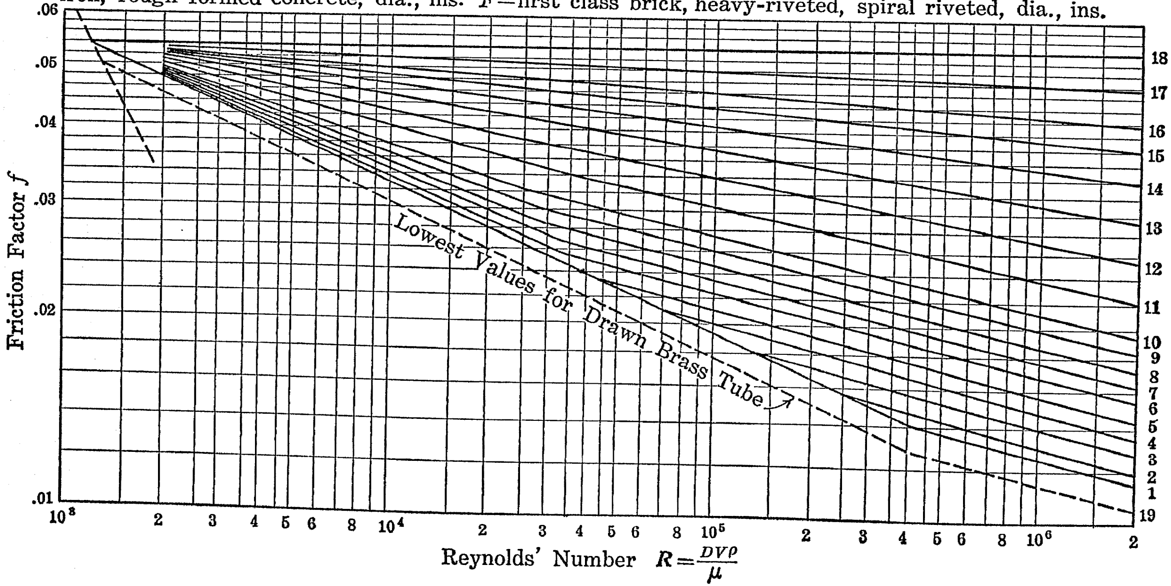


FIG. 119. Friction factor for pipes.

85. Classification and Solution of Problems Involving Pipe Friction. — Problems involving pipe flow in a given length of pipe may be classified according to the known data and the required results, as follows:

(1) Given the diameter, type of pipe and discharge, to find the head loss.

(2) Given the diameter, type of pipe, and allowable head loss, to find the discharge.

(3) Given the type of pipe, discharge and allowable head loss, to find the diameter.

Problems falling in classification (1) may be solved directly because with V, the diameter, and the physical properties of the fluid known, Reynolds number may be computed and f determined directly from Fig. 119. Then the use of the equation

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

will give the head loss in feet of fluid flowing.

Problems in group (2) are best solved by a trial and error method. In using the equation

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

for this type of problem, both f and V are unknown but there is a relationship between them as given by Fig. 119. The usual procedure is to assume a value of f. Ordinarily a value of about 0.02 is satisfactory for this first assumption. With this assumed value, an approximate velocity may be obtained from $h_f = f(L/D)(V^2/2g)$. With this value of V, a Reynolds number may be computed and a better value of f obtained from Fig. 119. A new value of V is determined, Reynolds number recomputed and the corresponding value of f compared with the value previously obtained. If the discrepancy between the last two values of f is not very great, say about 5%, the value of V previously obtained may be taken as the correct velocity and the discharge computed. Ordinarily the first value obtained from Fig. 119 will be found to be approximately the correct value.

Problems requiring the diameter of pipe for a given discharge and permissible head loss, group (3) above, are also best solved by trial and error. In this case, it is best to change the form of the Darcy equation to include the discharge, Q, instead of the velocity. Thus, since

$$Q = AV$$

$$V^2 = \left(\frac{Q}{A}\right)^2 = \frac{16Q^2}{\pi^2 D^4}$$

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

and

$$D^5 = \frac{16fLQ^2}{2g\pi^2} = \frac{fLQ^2}{39.7h_f} \tag{146}$$

In this equation, D, L, and h_f are in feet, and Q, is in cu. ft. per sec. With an assumed value of f, say 0.02, Eq. (146) can be solved for D and the corresponding velocity found from $Q = AV$. A Reynolds number is computed using this velocity and a new value of f determined from Fig. 119. D is then redetermined from Eq. (146). A standard commercial pipe with a diameter just larger than the one determined by the method indicated will prove satisfactory.

Illustrative problems of the preceding types are solved below.

Illustrative Problem 1: Find the head loss in 500 ft. of 4 in. average cast iron pipe discharging 1 c.f.s. of water at a temperature of 68° F.

This problem falls in group (1) above and can be solved directly.

$$V = \frac{Q}{A} = \frac{1}{0.0872} = 11.4 \text{ ft. per sec.}$$

The kinematic viscosity of water at 68° F. from Fig. 47 is

$$\nu = 1.09 \times 10^{-5} \text{ sq. ft. per sec.}$$

$$\mathbf{R} = \frac{DV}{\nu} = \frac{4 \times 11.4}{12 \times 1.09 \times 10^{-5}} = 349{,}000$$

From Fig. 119, using $\mathbf{R} = 349{,}000$ and curve 8,

$$f = 0.021$$

Then

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = 0.021 \times \frac{500}{\frac{1}{3}} \times \frac{\overline{11.4}^2}{64.4} = 63.6 \text{ ft.} \qquad \textit{Ans.}$$

Illustrative Problem 2: What is the discharge through an 8 in. clean steel pipe when oil having a specific gravity of 0.82 and an absolute viscosity of 6×10^{-3} lb. sec. per sq. ft. flows with a head loss of 20 ft. in 100 ft. of pipe?

This problem falls in group (2) and is best solved by trial and error.

Assume $f = 0.02$

Then from $$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

$$20 = 0.02\frac{100}{\frac{2}{3}}\frac{V^2}{2g}$$

$$V = \sqrt{\frac{20}{3} \times 64.4} = 20.7 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{DV\rho}{\mu} = \frac{2 \times 20.7 \times 0.82 \times 62.4}{3 \times 6 \times 10^{-3} \times 32.2} = 3650$$

From Fig. 119, using **R** = 3650 and curve 4,

$$f = 0.041$$

This value of f is much greater than the one originally assumed. It is, however, much closer to the correct value than 0.02. Using $f = 0.041$

$$20 = 0.041 \frac{100}{\frac{2}{3}} \frac{V^2}{2g}$$

$$V = \sqrt{\frac{20}{6.15} \times 64.4} = 14.5 \text{ ft. per sec.}$$

Since **R** is directly proportional to V, the new

$$\mathbf{R} = \frac{14.5}{20.7} \times 3650 = 2560$$

$$f = 0.0445 \text{ from Fig. 119}$$

One more trial gives

$$V = 13.9 \text{ ft. per sec.}$$

and

$$Q = 0.349 \times 13.9 = 4.85 \text{ c.f.s.}$$ *Ans.*

Illustrative Problem 3: Find the diameter of a riveted sheet duct required to deliver 500 cu. ft. per min. of air at atmospheric pressure and temperature of 60° F., if the allowable pressure drop is 3 in. of water in 100 ft. of duct. For air, $R = 53.3$. Assume that the pressure drop is relatively small.

From

$$pv = RT$$

$$v = \frac{RT}{p} \text{ and } w = \frac{p}{RT}$$

Therefore

$$w = \frac{14.7 \times 144}{53.3 \times (460 + 60)} = 0.076 \text{ lb. per cu. ft.}$$

The head loss permissible is 3 in. of water or

$$\frac{3 \times 62.4}{12} = 15.6 \text{ lb. per sq. ft.}$$

The head loss, in feet of fluid flowing, is then

$$h_f = \frac{15.6}{0.076} = 205 \text{ ft.}$$

Assuming $f = 0.02$, and substituting in Eq. (146) we have

$$D^5 = \frac{fLQ^2}{39.7h_f} = \frac{0.02 \times 100 \times (500)^2}{39.7 \times 205 \times (60)^2}$$

$$D^5 = 0.017$$

$$D = 0.442 \text{ ft.}$$

$$V = \frac{Q}{A} = \frac{500 \times 4}{60\pi \times (0.442)^2} = 54.4 \text{ ft. per sec.}$$

For air at this temperature and pressure

$$\nu = 1.5 \times 10^{-4} \text{ sq. ft. per sec.}$$

Then

$$\mathbf{R} = \frac{DV}{\nu} = \frac{0.442 \times 54.5}{1.5 \times 10^{-4}} = 161{,}000$$

From Fig. 119, using $\mathbf{R} = 161{,}000$ and curve 6,

$$f = 0.021$$

Since D^5 is directly proportional to f, in Eq. (146), the new

$$D^5 = \frac{0.021}{0.02} \times 0.017 = 0.0178$$

$$D = 0.446 \text{ ft. or } 5.35 \text{ in.}$$

A 6 in. diameter duct would be used. *Ans.*

PROBLEMS

170. Find the head loss in 300 ft. of 2 in. brass pipe when oil having a specific gravity of 0.85 and absolute viscosity of 2×10^{-3} lb. sec. per sq. ft. flows at the rate of 3 c.f.s.

171. Find the loss of head in 1000 ft. of 12 in. best cast iron pipe when 3 c.f.s. of water flow through the pipe at a temperature of 70° F.

172. Consider two pipes of the same size and material. Air at 100 lb. per sq. in. abs. flows in one while water flows in the other. The velocities are the same and the temperature is 80° F. Assume a certain type of pipe and velocity of flow, and find the ratio of the head loss in the water pipe to that in the air pipe. *Ans.* 0.89.

173. North Louisiana heavy crude oil is pumped through a 4 in. clean steel pipe a distance of 2 miles from the storage tanks to the loading wharf at the rate of 400 g.p.m. Find the lost head in this length of pipe. $T = 120°$ F.

174. Midcontinent residuum, 14.5 gravity, flows through a 2 in. clean steel pipe at the minimum turbulent velocity. Find the velocity of flow and the head loss per 100 ft. of pipe. $T = 130°$ F. *Ans.* $V = 132$ ft. per sec., $h_f = 7790$ ft.

175. Water flows in a 1 in. average galvanized pipe with a pressure drop of 10 lb. per sq. in. per 100 ft. of length. The temperature is 50° F. Find the discharge.

176. Oil with a kinematic viscosity of 1×10^{-4} sq. ft. per sec. and specific gravity = 0.82 flows through a 4 in. clean steel pipe with a pressure drop of 25 lb. per sq. in. in 300 ft. Find the discharge.

177. Carbon dioxide flows through a 3 in. wrought iron pipe at a temperature of 60° F. At one point in the line, the pressure is 60 lb. per sq. in. abs. At a point 300 ft. downstream, the pressure is 54 lb. per sq. in. abs. For carbon dioxide at this temperature $\mu = 3 \times 10^{-7}$ lb. sec. per sq. ft. and $R - 35.1$. Find the weight discharge.

178. Light motor oil, S.G. = 0.89 average, is pumped through a 3 in. wrought iron pipeline with a pressure drop of 10 lb. per sq. in. per 100 ft. of length. Find the discharge, (*a*) if the temperature is 60° F., (*b*) if the temperature is 120° F.

179. Find the diameter of an average cast iron pipe to discharge 1000 g.p.m. of water at 75° F. with a head loss of 8 ft. per 100 ft. of length.

180. Find the diameter of spiral riveted pipe required to discharge 5 c.f.s. of water at 60° F. with a head loss of 10 ft. per mile. *Ans.* $d = 18$ in.

181. Find the size of brass pipe required to discharge 200 cu. ft. per min. of carbon dioxide at an average pressure of 40 lb. per sq. in. abs. and temperature of 60° F. with an allowable pressure drop of 2 ft. of water in 200 ft. of pipe. The absolute viscosity of carbon dioxide at this temperature is $3.07 \times 10^{-}$ lb. sec. per sq. ft. and $R = 35.1$.

86. Flow in Non-Circular Sections. — Most problems of fluid flow in closed conduits involve pipes of circular cross section. In air duct or culvert design, square and rectangular cross sections are often used, and large sewers are often made elliptical in shape.

Experiments with *turbulent* flow in pipes indicate that the head loss depends upon a quantity called the hydraulic radius. This is defined as the area of the cross section of the stream divided by the wetted perimeter. The wetted perimeter is the length of the intersection of the fluid cross section and the wall of the conduit. Thus

$$R = \frac{A}{\text{w.p.}}$$

in which R is the hydraulic radius, A the area, and w.p. the wetted perimeter. The dimension of R is a length.

For circular pipes flowing full, the hydraulic radius is

$$R = \frac{\pi D^2}{4\pi D} = \frac{D}{4}$$

or

$$D = 4R$$

The equation derived for head loss in circular pipes can be used for non-circular sections by replacing D by $4R$. D in Reynolds number must also be replaced by $4R$ in using the Reynolds number-friction factor diagram.

The procedure outlined above for treating pipes of non-circular cross

section gives approximate results which are satisfactory for practical purposes when the flow is turbulent. For laminar flow, this approximation is not very close and actual tests to determine the relationship between f and **R** should be made.

PROBLEMS

182. Find the head loss in 50 ft. of concrete culvert 2 ft. square when discharging water at the rate of 5 c.f.s. Use $\nu = 1 \times 10^{-5}$ sq. ft. per sec.

183. How many cubic feet per minute of air are being discharged through a 10×15 in. sheet duct when the drop in pressure is 2 in. of water per 100 ft. The average pressure is 20 lb. per sq. in. abs. and the temperature is 60° F. $R = 53.3$.

87. Minor Losses. — It has been stated that the introduction of turbulence in uniform flow in a pipeline causes an increase in energy loss. This energy loss is traceable ultimately to the conversion of mechanical energy into heat through viscous action of the fluid. It follows, therefore, that any factor which increases the turbulence of a fluid will contribute to an increase in the amount of energy converted into heat. This has been demonstrated experimentally. Pipelines usually include a number of elements which, because of their shape, cause increased turbulence and consequently more head loss than the normal pipe friction loss. The losses arising from these elements are ordinarily termed *minor losses* because they are usually small in comparison with the friction loss in the pipe itself. This is not always the case, however, since the so-called minor losses may be responsible for a large proportion of the total energy loss.

Minor losses which may influence the flow in a pipeline are

(1) Loss due to an enlargement of the cross section.
(2) Loss at the entrance to the pipeline.
(3) Loss at exit.
(4) Loss due to a contraction of the cross section.
(5) Loss due to bends in the pipeline.
(6) Loss due to fittings such as valves, elbows, etc.

Expressions for the minor losses cannot, in general, be determined by analytical methods. However, these phenomena take place essentially in short lengths of pipe where viscosity plays no great role. The phenomenon is really one of inertia and the additional loss can be visualized as one of increased impact between fluid particles due to the more turbulent condition which results from the changed regimen of flow. The inertia properties, V and ρ, and not μ determine the loss. Referring to Eq. (74), Chap. VI, the pressure drop in a smooth pipe was given by

$$P = \frac{\rho V^2}{D}\phi(\mathbf{R})$$

Since our minor losses are independent of viscosity, our dimensional reasoning would give us

$$P = \text{Const.} \left(\frac{\rho V^2}{D}\right)$$

This reduces to

$$h_f = k\frac{V^2}{2g} \tag{147}$$

which is confirmed by experiment.

The individual losses are treated in the following paragraphs.

88. Loss Due to Enlargement of Section. — Enlargement from one cross section to a larger one may take place abruptly or gradually. The case of sudden enlargement will be considered first because direct application of the principles of mechanics and the Bernoulli equation lead to an equation which has close experimental verification. In addition, some of the other

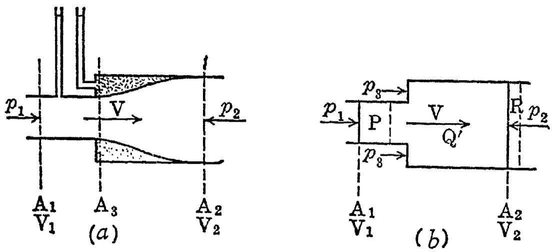


Fig. 120. The sudden enlargement.

minor losses have analogous characteristics, so that results obtained from an analysis of the sudden enlargement can be adapted for determining the other losses.

Figure 120*a* shows the conditions in a pipeline at a sudden enlargement from an area, A_1, to an area, A_2. A_2 is taken down stream far enough so that a steady regimen has been reestablished. At the corners of the large pipe, eddies are set up to which the enlargement loss may be largely attributed. Figure 120*b* is the free body diagram of the fluid between section (1) and (2) showing the pressure forces acting on the fluid in the direction of flow. The tangential stress at the wall of the pipe is relatively small and will be neglected. The magnitude of the pressure p_3 acting on an annular ring near the corner of the large pipe is usually assumed to be equal to p_1 for the following reason. Except for the eddying condition, the fluid in the corners is approximately at rest so that the pressure is taken to be

constant throughout this portion of the fluid. Directly at the enlargement, where the fluid issues from the small pipe, the pressure is p_1, consequently,

$$p_1 = p_3$$

Two piezometer tubes, attached to the pipe as shown in Fig. 120*a*, have shown p_1 and p_3 approximately equal.

The desired result is obtained by applying the two principles of mechanics to the mass of fluid between sections A_1 and A_2 in Fig. 120*b*:

(1) Conservation of energy as embodied in the Bernoulli equation.

(2) The resultant force equals the rate of change of momentum.

From the Bernoulli equation, we have

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + Z_2 + h_f$$

in which h_f is the loss due to the sudden enlargement. With the pipe horizontal,

$$Z_1 = Z_2$$

and

$$h_f = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \left(\frac{p_2}{w} - \frac{p_1}{w}\right) \tag{a}$$

The momentum principle states that the resultant force acting on a body is equal to the change in momentum of the body per second. Assuming that the change taking place in one second is shown by the two positions of the given mass $(P + Q')$ and $(Q' + R)$, the change in momentum is the difference between the momentum of R and that of P. The quantity of fluid in each of these portions is Q cu. ft. and the mass is wQ/g slugs. The rate of change in momentum towards the left is

$$\frac{wQ}{g}(V_1 - V_2) \tag{b}$$

in which V_1 and V_2 are the velocities at (1) and (2) respectively. The resultant force acting towards the left is

$$F = (p_2 - p_1)A_2 \tag{c}$$

Equating (b) and (c)

$$(p_2 - p_1)A_2 = \frac{wQ}{g}(V_1 - V_2) = \frac{w}{g}(A_2V_2)(V_1 - V_2)$$

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{V_2(V_1 - V_2)}{g}$$

Substituting this value of $p_2/w - p_1/w$ in (a), we have

$$h_f = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{V_2(V_1 - V_2)}{g}$$

$$= \frac{V_1^2 - V_2^2 - 2V_1V_2 + 2V_2^2}{2g}$$

$$= \frac{(V_1 - V_2)^2}{2g} \tag{148}$$

The reader should note that the numerator in Eq. (148) is the square of the difference of two velocities and not the difference of the squares.

Using the continuity principle in Eq. (148)

$$V_2 = \frac{A_1}{A_2} V_1$$

and

$$h_f = \frac{(V_1 - V_2)^2}{2g} = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{V_1^2}{2g} \tag{149}$$

or

$$h_f = k_1 \frac{V_1^2}{2g}$$

in which

$$k_1 = \left(1 - \frac{A_1}{A_2}\right)^2$$

and $V_1^2/2g$ is the larger *velocity head.*

In terms of the smaller *velocity head*, the loss is

$$h_f = \left(\frac{A_2}{A_1} - 1\right)^2 \frac{V_2^2}{2g} \tag{150}$$

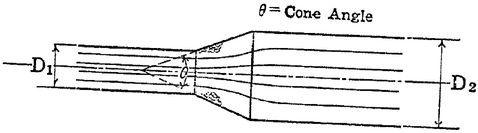


FIG. 121. The gradual enlargement.

When the enlargement takes place gradually, Fig. 121, the turbulence set up by the transition is less so that the energy loss from this cause is reduced. On the other hand for small cone angles, a considerable distance

exists between a section in the smaller pipe and the section of steady flow in the larger pipe. This tends towards a large pipe friction loss. For this reason, some cone angle should give a minimum value for the total loss due to the transition. Gibson has found experimentally that the cone angle for minimum total loss is about 6°; and that the greatest loss occurs with a cone angle of about 60°. For these values of cone angle, the loss is approximately 0.13 and 1.2 times that occurring with sudden enlargement, respectively. Gibson's curve showing the variation of loss in terms of the sudden enlargement loss as a function of cone angle is shown in Fig. 122.

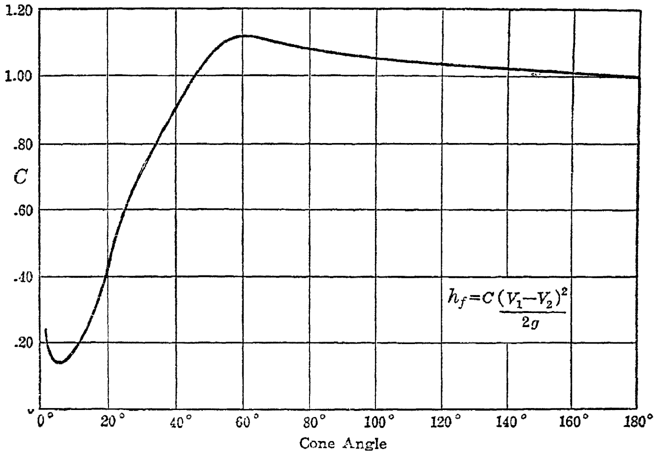


FIG. 122. Head loss in gradually enlarging circular pipes.

PROBLEMS

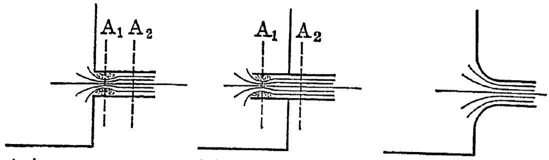
184. An 8 in. pipe is connected to a 12 in. pipe by means of a sudden enlargement. What is the loss of head due to the enlargement when 2 c.f.s. flow through the pipe? *Ans.* $h_f = 0.158$ ft.

185. A transition between a 6 in. pipe and 10 in. pipe is made by means of a section having a 45° cone angle. (*a*) Find the head loss in the transition when the discharge is 1 c.f.s. (*b*) How could the head loss in transition be reduced and what would be the minimum head loss?

186. The measured head loss caused by a sudden enlargement from a 2 in. to a 4 in. pipe is 0.44 ft. What is the rate of flow?

187. The diameter of a pipe is changed from 3 in. to 5 in. by means of a transition having a 30° cone angle. The measured loss of head due to the transition is 0.52 lb. per sq. in. If the fluid flowing has a specific gravity of 0.93, find the discharge.

89. Loss at Entrance. — The three common types of entrances to pipe lines are the square edged, the re-entrant, and the bell mouth. These are shown in Fig. 123. The flow conditions in each type of entry are also shown. It will be noticed that in both the square edged and re-entrant types, a contraction occurs followed by an enlargement. Conditions between the sections marked A_1 and A_2 are therefore quite similar to these occurring in



(*a*) Square Edge (*b*) Re-entrant (*c*) Bell Mouth

FIG. 123. Types of entrance.

sudden enlargements and the loss could be obtained by using Eq. (150) if the ratio of the pipe area to the contracted area were known. If the ratio of the contracted area to the pipe area is C_c, from Eq. (150), the loss in either case is

$$= \left(\frac{1}{C_c} - 1\right)^2 \frac{V^2}{2g} \tag{151}$$

in which V is the pipe velocity. It is generally assumed that the contraction for a square edged entry is similar to that for a standard orifice, or about 0.6. Thus for a square edged entry, the loss is

$$= \left(\frac{1}{0.6} - 1\right)^2 \frac{V^2}{2g} = 0.44 \frac{V^2}{2g}$$

This loss is usually taken as $0.5(V^2/2g)$.

For the re-entrant type, C_c is approximately the same as that for the Borda tube or 0.5. For this entry

$$h_f = \left(\frac{1}{0.5} - 1\right)^2 \frac{V^2}{2g} = \frac{V^2}{2g}$$

The bell-mouth entrance causes the least amount of energy loss. The fluid is directed gradually into the pipe and the turbulence, which is the fundamental cause of energy loss in the other types of entrance, is not present. Actually, it is an experimental fact that boundaries converging in the direction of flow tend towards more stable conditions of flow. The loss in a bell-mouth entrance is almost entirely due to normal friction. It

has been determined experimentally and is usually taken as

$$h_f = 0.05 \frac{V^2}{2g}$$

in which $V^2/2g$ is the velocity head in the pipeline.

90. Loss at Exit. — Conditions at the submerged exit end of a pipeline are shown in Fig. 124. Here the stream expands suddenly from the area of the pipe to an area which, for all practical purposes, is infinite. In the equation for sudden enlargement, Eq. (149), A_1/A_2 becomes zero and the loss is

$$h_f = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{V^2}{2g} = \frac{V^2}{2g}$$

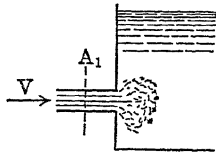


FIG. 124. Submerged exit.

91. Loss Due to Contraction. — Losses due to sudden contraction are usually expressed by the equation

$$h_f = k \frac{V_2^2}{2g}$$

in which k is an experimental coefficient depending on the ratio of the smaller diameter D_2 to the larger diameter, D_1, and $V_2^2/2g$ is the velocity head in the *smaller pipe*. Table VI gives average values of k for sudden contractions

TABLE VI

D_2/D_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k	0.46	0.44	0.42	0.38	0.34	0.28	0.21	0.14	0.06

92. Loss Due to Bends. — Bends in pipelines cause head losses greater than those occurring in equal lengths of straight pipe. This extra loss is due fundamentally to the increased turbulence arising from the change in direction of flow. Figure 125 shows a bend in a pipeline of diameter D. The radius of the bend is R and the angle of the bend θ.

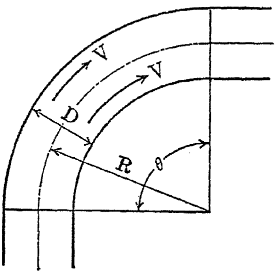


FIG. 125

At the entrance to the bend, the velocities in the cross section are those normal for turbulent flow in a straight pipe with a maximum velocity at the center. Immediately after the entrance to the bend, the direction of the velocities begins to change. For this change to take place, the pressure at the outside of the bend must be greater than that on the inside. Conditions tend to become similar to those for a free vortex and the velocities near the inside of

the bend become greater than those on the outside. This redistribution of velocity which, for the most part, takes place early in the bend, results in increased turbulence throughout the bend. Upon leaving the bend, conditions are reversed and the approximate free vortex motion established in the bend must be converted back to normal straight pipe flow. In addition to the approximate free vortex motion mentioned above, experimental investigation has disclosed a secondary circulatory motion in the plane of any cross section of the bend.

From the brief description of conditions at a bend given above, it may be concluded that the total excess loss of energy above that occurring in an equivalent length of straight pipe is made up of three parts:

(1) A loss at entrance to the bend due to a tendency for changing from rectilinear to vortex motion.

(2) A loss in the bend greater than normal pipe loss because of increased turbulence.

(3) An excess loss in the straight portion of the pipe following the bend due to re-establishment of normal pipe flow. This loss occurs over a considerable length of straight pipe beyond the bend.

The losses at entrance and exit seem to be independent of the angle θ. They depend upon the sharpness of the bend as measured by the ratio of the radius of the bend to the diameter of the pipe, R/D; the larger losses occurring with the smaller ratios. The loss in the bend itself is usually assumed to vary directly as the length of the bend. For any particular angle of bend the length along the centerline is directly proportional to the radius. Thus for small radii, the entrance and exit losses are comparatively large and the loss in the bend itself small. For large radii, the reverse is true. Consequently, there should be some ratio of R/D for which the total excess loss is least. Experiments show that this ratio of R/D is about 5.

The excess head loss in a bend is usually expressed by the equation

$$h_B = k\,\frac{V^2}{2g}$$

in which k is an experimental coefficient and V is the mean velocity in the pipe. Values of k as determined by Beij[1] for 90° bends in 4 in. steel pipes are shown in Fig. 126*a*. Results of other experimenters indicate that the loss due to bends is dependent upon the type of pipe and some unknown factors in addition to the R/D ratio. Agreement between the results of different investigators is lacking except in a qualitative sense. The losses due to bends are usually small, however, and Fig. 126*a* will give the coefficient without serious error.

[1] Beij, K. Hilding, "Pressure Losses for Fluid Flow in 90° Pipe Bends." *Research Paper RP* 1110, *National Bureau of Standards*, July, 1938.

The total excess loss in bends is not directly proportional to the angle of the bend. The entrance and exit losses remain approximately constant for the same R/D ratio while the loss in the bend itself is about propor-

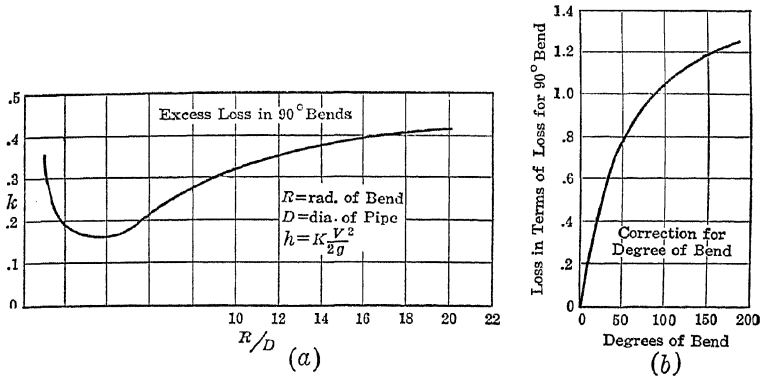


FIG. 126. Bend losses.

tional to the length along the centerline. Thus the loss in a 90° bend is not equal to twice that in a 45° bend. The curve shown in Fig. 126*b* gives the loss in bends in terms of the loss in a 90° bend of the same R/D ratio.

Illustrative Problem: Find the total head loss occurring in a 90° bend in a 6 in. steel pipe discharging 1.53 c.f.s. of water at 60° F. The radius of the bend is 4 ft.

$$V = \frac{Q}{A} = \frac{1.53}{0.196} = 7.81 \text{ ft. per sec.}$$

Centerline length of 90° bend with 4 ft. radius is

$$L = 2\pi R \quad 2\pi \times 4 = 6.28 \text{ ft.}$$

Reynolds number

$$\mathbf{R} = \frac{DV}{\nu} = \frac{1 \times 7.81}{2 \times 1.22 \times 10^{-5}} = 320{,}000$$

From curve 4, Fig. 119,

$$f = 0.017$$

Normal pipe friction loss

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = 0.017 \times \frac{6.28}{\frac{1}{2}} \times \frac{\overline{7.81}^2}{2g} \quad 0.20 \text{ ft.}$$

$$\frac{R}{D} \quad \frac{4}{\frac{1}{2}}$$

The excess loss coefficient from Fig. 126*a* is

$$k = 0.27$$

Excess loss due to bend

$$h_B = k\frac{V^2}{2g} = \frac{0.27 \times 7.81^2}{2g} = 0.255 \text{ ft.}$$

Total loss due to bend

$$h_f + h_B = 0.200 + 0.255 = 0.455 \text{ ft.} \qquad \textit{Ans.}$$

PROBLEMS

188. Find the total loss in the preceding illustrative problem if the change in direction were accomplished by two 45° bends of the same radius spaced a considerable distance apart.

189. (*a*) Find the approximate radius for minimum head loss in a 180° bend in 4 in. steel pipe. (*b*) Find the total head loss in the above bend for a discharge of 2 c.f.s. of oil having a kinematic viscosity of 2×10^{-5} sq. ft. per sec.

190. What is the excess loss of head due to a 90° bend in a 12 in. sheet metal duct having a radius of 10 ft., and discharging 600 cu. ft. per min. of air a temperature of 60° F. and a pressure of 20 lb. per sq. in. abs.? Express your answer in inches of water. *Ans.* $h_f = 0.0156$ in.

93. Losses in Fittings and Valves. — The loss of head in fittings and valves is expressed as the loss in a given length of pipe in pipe diameters, or as some coefficient times the velocity head. The latter method seems preferable since it lends itself more readily to solution.

Values of k in $h_f = kV^2/2g$ for losses occurring in the more common types of valves and fittings have been experimentally determined. Table VII gives the valves as compiled by the Crane Co. from the results of their tests and from the tests of others.

TABLE VII — COEFFICIENT k FOR MINOR LOSSES IN VALVES AND FITTINGS

Type	k
Globe valve	10.0
Angle valve	5.0
Close return bend	2.2
Standard tee	1.8
Short radius elbow	0.9
Medium radius elbow	0.75
Long radius elbow	0.60
45° elbow	0.42
Gate valve (fully open)	0.19

Courtesy: Crane Co.

94. Summary of Coefficients for Minor Losses. — A summary of the coefficients for the more common minor losses occurring in pipeline prob-

lems is given in Fig. 127. This, in connection with Fig. 119, will enable the student to solve most of the problems dealing with pipe flow.

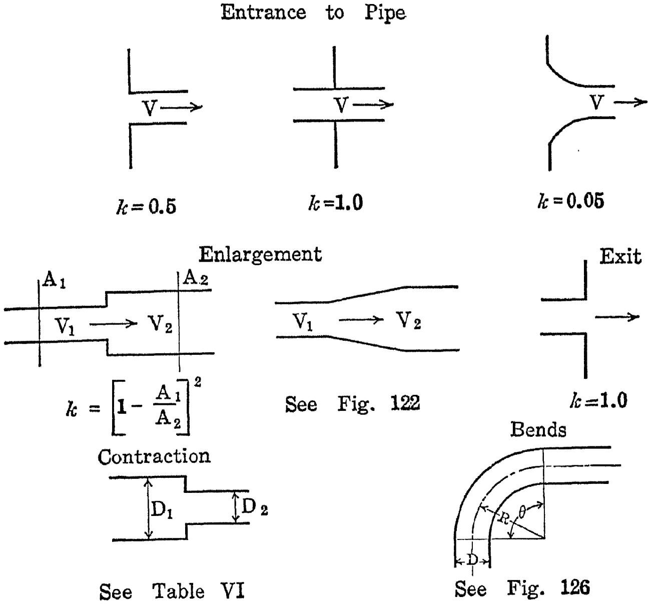


FIG. 127

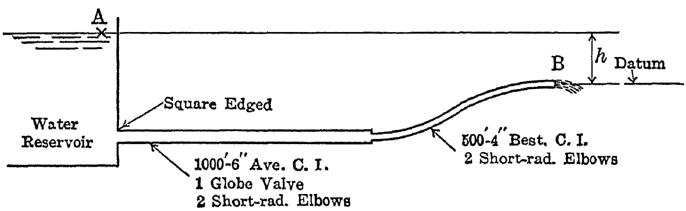


FIG. 128

95. Pipe Problems Including All Losses. — Pipelines often include all, or most of, the minor losses discussed in the preceding articles. When the minor losses are small in comparison with the pipe friction loss they can usually be neglected.

Problems involving minor losses may be classified in the same manner used in Art. 85 for problems involving pipe friction only. The following illustrative problems should clarify the method of solution for each type.

Illustrative Problem 1: Fig. 128 shows a pipe leading from a large reservoir. The discharge through the pipe is 1 c.f.s. at a temperature of 60° F. Find the eleva-

tion of the surface of the reservoir above the outlet of the pipe: (*a*) considering all losses; (*b*) neglecting minor losses.

This problem can be solved directly. Choose a datum plane through the end of the pipe and write Bernoulli's equation between the surface of the reservoir and the end of the pipe. In order to distinguish between terms applying to the different parts of the pipeline, the diameter of the pipe will be used as a subscript. Thus, V_6 is the velocity in the 6 in. pipe, f_6 is the friction factor in the 6 in. pipe, etc.

p_A/w and p_B/w are both atmospheric or zero; $V_A^2/2g$ is zero because the reservoir is considered large; $V_B^2/2g$ is the velocity head in the 4 in. pipe; and Z_B is zero. h_f is the total head loss between A and B, and includes:

(1) Loss at entrance $= 0.5\dfrac{V_6^2}{2g}$

(2) Loss in globe valve $= 10\dfrac{V_6^2}{2g}$

(3) Friction loss in 6 in. pipe $= f_6\dfrac{L_6}{D_6}\dfrac{V_6^2}{2g} = f_6\dfrac{1000}{\frac{1}{2}} \times \dfrac{V_6^2}{2g}$

(4) Loss in 2 short radius 6 in. elbows $= 2 \times 0.9\dfrac{V_6^2}{2g}$

(5) Loss at contraction $= 0.23\dfrac{V_4^2}{2g}$

(6) Loss in 4 in. pipe $= f_4\dfrac{L_4}{D_4}\dfrac{V_4^2}{2g} = f_4\dfrac{500}{\frac{1}{3}} \times \dfrac{V_4^2}{2g}$

(7) Loss in 2 short radius 4 in. elbows $= 2 \times 0.9\dfrac{V_4^2}{2g}$

The friction factors are obtained from Fig. 119 after the value of **R** has been determined. At 60° F., ν for water is 1.22×10^{-5} sq. ft. per sec. $V_6 = 5.1$ ft. per sec. and $V_4 = 11.45$ ft. per sec.

$$\mathbf{R}_6 = \frac{5.1}{2 \times 1.22 \times 10^{-5}} = 209{,}000$$

$$\mathbf{R}_4 = \frac{11.45}{3 \times 1.22 \times 10^{-5}} = 313{,}000$$

From curve 7, Fig. 119, $f_6 = 0.0215$ and $f_4 = 0.0205$.

Upon substituting these values in Bernoulli's equation, Eq. (a) follows:

$$0 + 0 + h = 0 + \frac{V_4^2}{2g} + 0 + \left(0.5 + 10 + 2 \times 0.9 + 0.0215 \times \frac{1000}{\frac{1}{2}}\right)\frac{V_6^2}{2g}$$
$$+ \left(0.23 + 2 \times 0.9 + 0.0205 \times \frac{500}{\frac{1}{3}}\right)\frac{V_4^2}{2g}. \quad \text{(a)}$$

Using the equation of continuity,

$$\frac{V_6^2}{2g} = \left(\frac{D_4}{D_6}\right)^4 \frac{V_4^2}{2g} = \frac{16}{81}\frac{V_4^2}{2g}$$

$$h = (12.3 + 43)\frac{V_6^2}{2g} + (3.0 + 30.8)\frac{V_4^2}{2g}$$

$$\left[(55.3)\left(\frac{16}{81}\right) + 33.8\right]\frac{V_4^2}{2g} = 44.7\frac{V_4^2}{2g}$$

$$44.7\frac{(11.45)^2}{2g} = 91.0 \text{ ft.} \qquad \textit{Ans.}$$

The reader will note that a velocity head, in this case $V_4^2/2g$, was retained as a common factor until the final step in the solution. By so doing, the numerical work was considerably shortened.

The solution for part (b) may be obtained quite simply by neglecting all the terms representing minor losses in (a). Considering the velocity head at the end of the pipe and pipe friction only,

$$h = \frac{V_4^2}{2g} + 43\frac{V_6^2}{2g} + 30.8\frac{V_4^2}{2g}$$

$$= (1 + 8.5 + 30.8)\frac{V_4^2}{2g}$$

$$= 40.3\frac{(11.45)^2}{2g} = 82.0 \text{ ft.} \qquad \textit{Ans.}$$

Illustrative Problem 2: The pipeline shown in Fig. 129 discharges water at 60° F. (*a*) Find the discharge through the system when the pressure gage at *A* registers 100 lb. per sq. in. (*b*) Find the pressure in the 3 in. pipe at *B*.

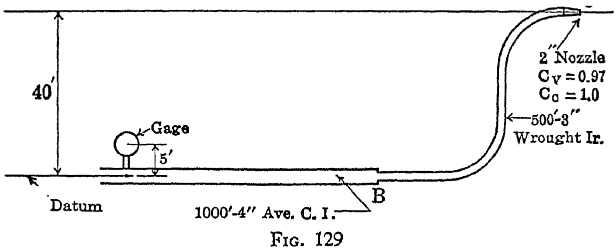


FIG. 129

Choose a datum plane through *A* and write Bernoulli's equation between *A* and *C*.

$$\frac{p_A}{w} \quad \frac{100 \times 144}{62.4} + 5 = 231 + 5 = 236 \text{ ft.}$$

$\dfrac{V_A^2}{2g}$ is the velocity head in 4 in. pipe $= \dfrac{V_4^2}{2g}$

$Z_A = 0, \dfrac{p_c}{w} = 0, Z_c \quad 40$ ft.

$\dfrac{V_c^2}{2g}$ is the velocity head at the end of the 2 in. nozzle $= V_2^2/2g$

h_f is the loss between A and C and includes:

(1) Friction loss in 4 in. pipe $= f_4 \dfrac{L_4}{D_4} \dfrac{V_4^2}{2g}$

(2) Loss at contraction $= 0.18 \dfrac{V_3^2}{2g}$

(3) Friction loss in 3 in. pipe $= f_3 \dfrac{L_3}{D_3} \dfrac{V_3^2}{2g}$

(4) Friction loss in nozzle $= \left(\dfrac{1}{C_v^2} - 1\right) \dfrac{V_2^2}{2g}$

$$\frac{V_4^2}{2g} = \frac{1}{16} \frac{V_2^2}{2g}; \quad \frac{V_3^2}{2g} = \frac{16}{81} \frac{V_2^2}{2g}$$

Since the velocity is unknown, $\mathbf{R}$ is unknown and a value of f must be assumed. Assume $f_4 = f_3 = 0.02$. Substituting in the Bernoulli equation,

$$236 + \frac{V_4^2}{2g} = \frac{V_2^2}{2g} + 40 + \left(0.02 \times \frac{1000}{\frac{1}{3}}\right) \frac{V_4^2}{2g} + \left(0.18 + 0.02 \times \frac{500}{\frac{1}{4}}\right) \frac{V_3^2}{2g}$$
$$+ \left(\frac{1}{(0.97)^2} - 1\right) \frac{V_2^2}{2g}$$

Then

$$196 = 59 \frac{V_4^2}{2g} + 40.2 \frac{V_3^2}{2g} + 1.06 \frac{V_2^2}{2g}$$
$$= (3.69 + 7.94 + 1.06) \frac{V_2^2}{2g}$$
$$= 12.69 \frac{V_2^2}{2g}$$
$$V_2 = \sqrt{2g\left(\frac{196}{12.69}\right)} = 31.5 \text{ ft. per sec.}$$
$$V_4 = \frac{V_2}{4} = 7.88 \text{ ft. per sec.}$$
$$V_3 = \frac{4V_2}{9} = 14.0 \text{ ft. per sec.}$$

The assumed values of f may now be checked.

$$\mathbf{R}_4 = \frac{D_4 V_4}{\nu} = \frac{7.88}{3 \times 1.22 \times 10^{-5}} = 215{,}000$$
$$\mathbf{R}_3 = \frac{D_3 V_3}{\nu} = \frac{14.0}{4 \times 1.22 \times 10^{-5}} = 287{,}000$$

From curve 8, Fig. 119, $f_4 = 0.0227$

From curve 6, Fig. 119, $f_3 = 0.0194$

Rewriting Bernoulli's equation

$$236 + \frac{V_4^2}{2g} = \frac{V_2^2}{2g} + 40 + (0.0227 \times 3000)\frac{V_4^2}{2g} + (0.18 + 0.0194 \times 2000)\frac{V_3^2}{2g} + 0.06\frac{V_2^2}{2g}$$

$$196 = 67.1\frac{V_4^2}{2g} + 39.0\frac{V_3^2}{2g} + 1.06\frac{V_2^2}{2g}$$

$$= (4.20 + 7.70 + 1.06)\frac{V_2^2}{2g}$$

$$12.96\frac{V_2^2}{2g}$$

$$V_2 = \sqrt{2g\left(\frac{196}{12.96}\right)} = 31.2 \text{ ft. per sec.}$$

This value of velocity is nearly the same as the one originally computed, and further correction is unnecessary. This is not always the case, however, and another determination of the friction factors might be necessary.

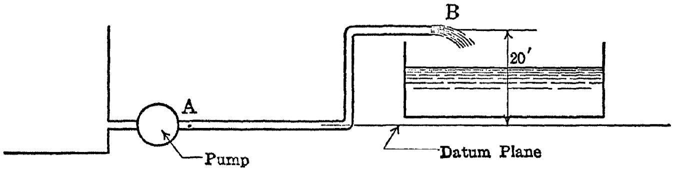


Fig. 130

The discharge may now be obtained

$$Q = A_2V_2 = 0.0218 \times 31.2 = 0.68 \text{ c.f.s.} \qquad \textit{Ans.}$$

The pressure at B is found by applying the Bernoulli equation between A and B.

$$236 + \frac{V_4^2}{2g} = \frac{p_B}{w} + \frac{V_3^2}{2g} + (0.0227 \times 3000)\frac{V_4^2}{2g} + 0.18\frac{V_3^2}{2g}$$

$$\frac{p_B}{w} = 236 - 67.1\frac{V_4^2}{2g} - 1.18\frac{V_3^2}{2g}$$

$$= 236 - 63.5 - 3.5 = 169 \text{ ft.}$$

$$p_B = 169 \times \frac{62.4}{144} = 73.1 \text{ lb. per sq. in.} \qquad \textit{Ans.}$$

Illustrative Problem 3: The pump shown in Fig. 130 has a capacity of 2 c.f.s. against a pressure of 60 lb. per sq. in. at the discharge side. The fluid is oil with a

specific gravity of 0.85 and kinematic viscosity of 1×10^{-4} sq. ft. per sec. The oil is to be delivered at a point 20 ft. above the center of the pump. The pipeline is 400 ft. long and contains a gate valve and 2 medium radius elbows. Find the size of average cast iron pipe required.

$$\frac{p_A}{w} = \frac{60 \times 144}{0.85 \times 62.4} = 162 \text{ ft. of oil}$$

$$Z_A = 0; \frac{p_B}{w} = 0; \ Z_B = 20; \frac{V_A^2}{2g} \quad \frac{V_B^2}{2g}$$

The friction loss between A and B includes:

(1) Loss at gate valve $= 0.19 \dfrac{V^2}{2g}$

(2) Friction loss in 400 ft. of average cast iron pipe $= f \dfrac{400}{D} \dfrac{V^2}{2g}$

(3) Loss in 2 medium radius elbows $= 2 \times 0.75 \dfrac{V^2}{2g}$

Substituting in the Bernoulli equation,

$$162 + \frac{V^2}{2g} + 0 = 0 + \frac{V^2}{2g} + 20 + \left(0.19 + 2 \times 0.75 + f\frac{400}{D}\right)\frac{V^2}{2g}$$

$$\left(0.19 + 2 \times .75 + \frac{f\,400}{D}\right)\frac{V^2}{2g} = 142$$

Now

$$V = \frac{Q}{A} = \frac{4Q}{\pi D^2}$$

$$\frac{V^2}{2g} \quad \frac{16Q^2}{\pi^2 D^4 \times 2g}$$

Therefore

$$\left(1.69 + \frac{f\,400}{D}\right)\frac{16Q^2}{\pi^2 D^4 \times 2g} = 142$$

$$Q = \sqrt{\frac{2g\pi^2 D^5 \times 142}{16(1.69D + 400f)}} \quad \sqrt{\frac{5650D^5}{1.69D + 400f}} = 2 \text{ c.f.s.} \qquad \text{(a)}$$

Since f and D are both unknown, the solution is best accomplished by assuming values of D. The velocity and a Reynolds number may then be computed and f obtained from Fig. 119. These tentative values of D and f are then substituted in the above equation. If the equation is satisfied, the assumed value of D was correct; if not, other values of D are assumed until the equation is satisfied. Thus, assuming a 6 in. diameter

$$V = \frac{2}{0.196} = 10.2 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{DV}{\nu} \quad \frac{6 \times 10.2}{12 \times 1 \times 10^{-4}} = 51{,}000$$

f from curve 7, Fig. 119, $= 0.025$

Substituting in (a)

$$\sqrt{\frac{5650 \times 0.5^5}{1.69 \times \frac{1}{2} + 400 \times 0.025}} = 4.04 \text{ c.f.s.}$$

This discharge is larger than that required, and a 6 in. diameter is too great. Equation (a) shows that Q is approximately proportional to $D^{5/2}$ if minor losses and variations in f are neglected. This relationship enables us to choose a new trial diameter for the required discharge. Thus

$$\frac{D^{5/2}}{(\frac{1}{2})^{5/2}} = \frac{2}{4.04}$$

$$D = \left(\frac{2}{4.04}\right)^{2/5} \times \tfrac{1}{2} = 0.755 \times \tfrac{1}{2} = 0.378 \text{ ft. or } 4.5 \text{ in.}$$

Assuming a 4.5 in. pipe

$$V = \frac{2}{0.112} = 17.8 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{4.5 \times 17.8}{12 \times 1 \times 10^{-4}} = 66{,}800$$

$$f \text{ from curve 8, Fig. 119, } = 0.026$$

Substituting in Eq. (a)

$$\sqrt{\frac{5650}{\left(1.69 \times \frac{4.5}{12} + 0.026 \times 400\right)} \times \left(\frac{4.5}{12}\right)^5} = \sqrt{\frac{5650}{11.0} \times 0.00755} = 1.97 \text{ c.f.s.}$$

If the required discharge of 2 c.f.s. is rigid, a 5 in. pipe would be required. Should there be a slight leeway, the $4\frac{1}{2}$ in. pipe would be satisfactory. *Ans.*

PROBLEMS

191. A 12 in. average cast iron pipe carries water between two reservoirs 1000 ft. apart. The line contains a square-edged entry, a gate valve and two medium radius elbows. The outlet is submerged. Find the difference in elevation between the surfaces of the reservoirs when the discharge is 3 c.f.s. $\nu = 1 \times 10^{-5}$ sq. ft. per sec.

192. A pump is used to deliver 2500 gal. per min. of oil to a large storage tank through an 8 in. clean steel pipe 400 ft. long. The elevation of the surface of the oil in the storage tank is 20 ft. above the delivery end of the pump. The pipeline includes a gate valve, 3 short radius elbows, and the outlet is submerged. S.G. = 0.87 and $\nu = 4 \times 10^{-5}$ sq. ft. per sec. Find the pressure at the delivery end of the pump. *Ans.* p = 27.5 lb. per sq. in.

193. Water at 60° F. discharges at the rate of 1 c.f.s. from a reservoir through 500 ft. of 6 in. average cast iron pipe followed by 200 ft. of 4 in. average cast iron pipe which terminates with a 2 in. nozzle for which $C_c = 1.0$ and $C_v = 0.95$. The inlet is re-entrant. (*a*) Find the elevation of the water in the reservoir. (*b*) Find the pressure at a point immediately after the reduction in pipe diameter if this point is 50 ft. below the surface of the reservoir.

194. A horizontal pipeline, discharging into the atmosphere, leads from a reservoir containing water and consists of 1000 ft. of 8 in. wrought iron pipe. A globe valve and 6 short radius elbows are in the line. Find the discharge through the pipeline when the elevation in the reservoir is 200 ft. above the end of the pipe. Assume a square-edged entry and $\nu = 1 \times 10^{-5}$ sq. ft. per sec.

195. Oil with a kinematic viscosity of 3×10^{-5} sq. ft. per sec. and a specific gravity of 0.92 is being pumped through a pipeline consisting of a 500 ft. of 6 in. clean steel pipe followed by 400 ft. of 4 in. clean steel pipe. The 6 in. pipe includes a globe valve and 3 medium radius elbows. The 4 in. pipe has 2 short radius elbows. The two pipes are connected by a sudden contraction and the discharge is into the atmosphere 40 ft. above a point near the beginning of the pipeline where the pressure is 100 lb. per sq. in. Find the discharge.

196. A 2 ft. square concrete culvert, 100 ft. long, carries water underneath a roadway. Find the discharge through the culvert when both ends are submerged and the difference in the elevation of the water surface on opposite sides of the road is 3 ft. Use $\nu = 1 \times 10^{-5}$ sq. ft. per sec. and k for entrance equal to 0.3.

Ans. $Q = 39.2$ c.f.s.

197. The pressure drop in 100 ft. of 16 in. diameter sheet metal air duct is 2 in. of water. The duct includes $1-180°$ and $4-90°$ bends having 3 ft. radii. The duct discharges into the atmosphere. $T = 70°$ F. and $\nu = 1.6 \times 10^{-4}$ sq. ft. per sec. Find the discharge in cubic feet per minute.

198. The pressure at one point in a 2 in. wrought iron pipe carrying methane at a temperature of 100° F. is 20 lb. per sq. in. abs. At another point 200 ft. downstream, the pressure is 19 lb. per sq. in. abs. The length of pipe includes a globe valve and 4 close return bends. Find the weight flow. ν for methane at this temperature and pressure is 2×10^{-4} sq. ft. per sec. $R = 96.5$.

199. Two water reservoirs are connected by a concrete pipe 600 ft. long with both ends re-entrant and submerged. The discharge through the pipe is 15 c.f.s. when the difference in the elevation of the surfaces of the reservoirs is 4 ft. Find the diameter of the pipe. $\nu = 1.2 \times 10^{-5}$ sq. ft. per sec.

200. Find the required diameter of an average cast iron pipe, 200 ft. long, to deliver 2 c.f.s. of water at 75° F. to a point 50 ft. below the surface of the reservoir. The inlet to the pipe is re-entrant and the line includes a gate valve and 2 medium radius elbows. *Ans.* $d = 5$ in.

96. Hydraulic Gradient. Total Energy Gradient. — In the design of pipelines, it is often necessary to study the variation of pressure along the pipe. Excessive pressures may cause the pipe to burst. Large negative pressures are also dangerous for two reasons:

(1) The greater outside atmospheric pressure may cause the pipe to collapse.

(2) Air may collect at points of negative pressure because of leaks and cause a serious obstruction to the flow.

Conditions along a pipeline are best studied by means of the hydraulic gradient or hydraulic grade line. The hydraulic gradient is the graphical representation of the quantity $(p/w) + Z$ for every point in the pipeline. Since Z is the elevation of a point in the pipeline above the datum plane, the distance between the pipeline and the hydraulic gradient is the pressure head, p/w, in the pipe at the point in question, and would be the height

to which the fluid would rise above the pipe in a vertical tube connected to it at the point in question. Thus the hydraulic gradient is often defined as a line connecting the points to which the fluid would rise in vertical piezometer tubes attached to the pipe.

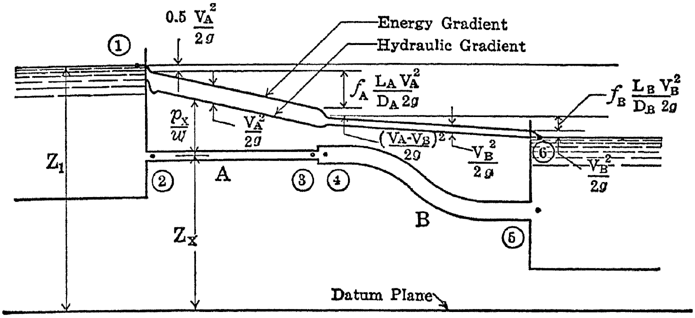


FIG. 131

The total energy gradient is the graphical representation of the total head at any point along the pipeline above some datum plane. Since the total energy per pound of fluid is the sum of the pressure, velocity and elevation heads, the energy gradient is a distance

$$H = \frac{p}{w} + \frac{V^2}{2g} + Z$$

above the datum plane. The vertical drop in the energy gradient between any two points (1) and (x) is, from the Bernoulli equation,

$$\left(\frac{p_1}{w} + \frac{V_1^2}{2g} + Z_1\right) - \left(\frac{p_x}{w} + \frac{V_x^2}{2g} + Z_x\right) = h_{f_{1-x}}$$

Thus, if one point on the total energy gradient is known, the elevation of another point on the line is found by subtracting the loss of head between the points from the known elevation.

Since the total energy gradient is a distance $\left(\frac{p}{w} + \frac{V^2}{2g} + Z\right)$ above the datum plane and the hydraulic gradient is $\left(\frac{p}{w} + Z\right)$ above the datum, the hydraulic gradient is a distance equal to $V^2/2g$ below the energy gradient. The hydraulic gradient, therefore, is most easily plotted by first establishing the energy gradient and then plotting the hydraulic gradient a distance $V^2/2g$ below it.

Construction of the hydraulic gradient will be further explained by considering an actual problem. Figure 131 shows a pipeline connecting

two reservoirs. It consists of two pipes of different diameters. The critical points in the line have been numbered. We assume that the discharge and friction factors are known. For point (1),

$$\frac{p}{w} = 0, \quad Z = Z_1, \quad \frac{V_1^2}{2g} = 0$$

Therefore

$$\frac{p}{w} + \frac{V^2}{2g} + Z = Z_1$$

and the energy gradient starts at the water surface. Between points (1) and (2), an entry loss occurs equal to 0.5 $(V_A^2/2g)$ for the square-edged entry shown. Therefore, the energy gradient at (2) is 0.5 $(V_A^2/2g)$ below the surface of the reservoir. Between (2) and (3) a friction loss takes place equal to $f_A \frac{L_A}{D_A} \frac{V_A^2}{2g}$. This loss occurs uniformly along the axis of the pipe. At the enlargement, an expansion loss occurs so that the total energy gradient drops an amount equal to this loss. Between (4) and (5), a friction drop equal to $f_B \frac{L_B}{D_B} \frac{V_B^2}{2g}$ takes place. Between (5) and (6), the energy gradient drops due to the exit loss and thereby reaches the surface of the second reservoir.

The hydraulic gradient can now be plotted below the energy gradient by an amount equal to the velocity head in the pipe at that point.

The reader should note that there is a continual drop in the energy gradient but that the hydraulic gradient may rise where the velocity decreases due to a change in section.

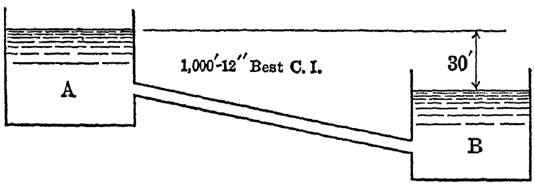


FIG. 132

PROBLEMS

201. Water is flowing through the 12 in. pipe shown in Fig. 132. (*a*) Find the discharge. (*b*) Sketch the energy gradient showing significant values plotted below a horizontal line at the surface of reservoir *A*. (*c*) Sketch the hydraulic gradient. $\nu = 1 \times 10^{-5}$ sq. ft. per sec.

202. The head, h, in Fig. 133 is such that the discharge through the pipeline is 2.24 c.f.s. Sketch the energy and hydraulic gradients showing significant values. $\nu = 1.2 \times 10^{-5}$ sq. ft. per sec.

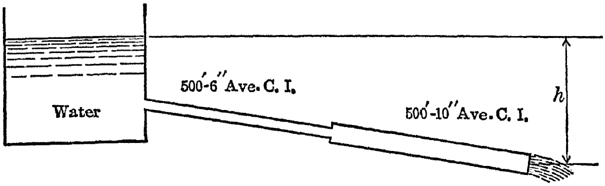


FIG. 133

203. Sketch the energy and hydraulic gradients for Illustrative Problem (1), Art. 95.

204. Sketch the energy and hydraulic gradients for Illustrative Problem (2), Art. 95.

205. Sketch the energy and hydraulic gradients for Illustrative Problem (3), Art. 95.

97. Pipes in Parallel. — Two pipes are connected in parallel as shown in Fig. 134. In this kind of a system, the flow must adjust itself so the head loss, h_f, for each pipe is the same.

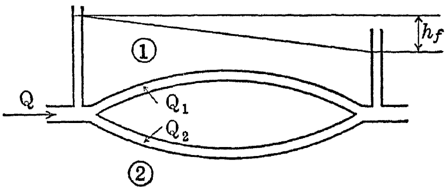


FIG. 134

Two types of problems may arise:

(1) Given the dimensions and kind of pipe, and the allowable head loss h_f, to find the discharge through the system.

(2) Given the total discharge, Q, through the system, and the kind and size of pipe, to find Q_1 and Q_2, and the head loss h_f.

Ordinarily the minor losses for each pipe are small and may be neglected. In this case,

$$h_f = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} = f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} \tag{a}$$

When h_f is known, type (1), each pipe may be treated separately. Consistent values of f and V for each pipe in (a) are determined, and the discharge for each pipe computed. Then

$$Q = Q_1 + Q_2$$

The following discussion should make clear the method of solution for problems of type (2).

If the fraction of the total flow through each pipe for *any* flow through the system is determined then approximately the same proportion of the

actual flow will take place in each pipe under the given condition. The steps for determining the ratio of the flow in each pipe to the total flow are:

(1) Assume a discharge Q_1' in one of the pipes.

(2) Compute the head loss in this pipe.

(3) With this head loss, determine Q_2' in the other pipe. Then the total discharge under the assumed conditions is

$$Q' = Q_1' + Q_2'$$

and Q_1'/Q' is the fraction of total flow in pipe (1) and Q_2'/Q' the fraction of the total flow in pipe (2). With these ratios known, the approximate discharges for the given flow will be

$$Q_1 = \frac{Q_1'}{Q'} Q$$

$$Q_2 = \frac{Q_2'}{Q'} Q$$

A final check on the accuracy of the answers may be obtained by computing the head loss for each pipe with the corrected discharges. These head losses should be approximately the same.

Illustrative Problem: Pipe (1) in Fig. 134 consists of 1000 ft. of 6 in. average cast iron pipe; pipe (2) is 2000 ft. of 8 in. best cast iron. The total discharge through the system is 2 c.f.s. Find the discharge through each pipe and the head loss. Water is flowing with $\nu = 1 \times 10^{-5}$ sq. ft. per sec. Assume

$$Q_1' = 1 \text{ c.f.s. } V_1 = \frac{1}{0.196} = 5.1 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{DV}{\nu} = \frac{6 \times 5.1}{12 \times 1 \times 10^{-5}} = 255{,}000$$

From curve 7, Fig. 119, $f = 0.021$

$$h_f = f_1 \frac{L_1}{D_1}\frac{V^2}{2g} = 0.021 \times \frac{1000}{\frac{1}{2}} \times \frac{\overline{5.1}^2}{2g} = 17.0 \text{ ft.}$$

Assuming $f = 0.02$ for the 8 in. pipe

$$17 = 0.02 \times \frac{2000}{\frac{2}{3}}\frac{V_2^2}{2g}$$

$$V_2 = \sqrt{\frac{2 \times 17 \times 2g}{3 \times 0.02 \times 2000}} = 4.27 \text{ ft. per sec.}$$

Check f

$$\mathbf{R} = \frac{DV}{\nu} = \frac{2 \times 4.27}{3 \times 1 \times 10^{-5}} = 285{,}000$$

From curve 6, Fig. 119, $f = 0.0196$ (Satisfactory)

$$Q_2' = A_2V_2 = 0.349 \times 4.27 = 1.49 \text{ c.f.s.}$$

$$Q' = Q_1' + Q_2' = 1 + 1.49 = 2.49 \text{ c.f.s.}$$

$$Q_1 = \frac{Q_1'}{Q'}\,Q = \frac{1}{2.49} \times 2 = 0.80 \text{ c.f.s.} \qquad \textit{Ans.}$$

$$Q_2 = Q - Q_1 = 2 - 0.80 = 1.20 \text{ c.f.s.} \qquad \textit{Ans.}$$

The new velocity in (1) is 4.08 ft. per sec.

$$\mathbf{R} = \frac{DV}{\nu} = \frac{1 \times 4.08}{2 \times 1 \times 10^{-5}} = 204{,}000$$

From curve 7, Fig. 119, $f = 0.0213$

$$h_f = f_1 \frac{L_1}{D_1}\frac{V_1^2}{2g} = 0.0213 \times \frac{1000}{\frac{1}{2}} \times \frac{\overline{4.08}^2}{2g} = 11.0 \text{ ft.} \qquad \textit{Ans.}$$

As a check, the head loss in the other pipe will be computed,

$$V_2 = \frac{Q_2}{A_2} = \frac{1.20}{0.349} = 3.44 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{DV}{\nu} = \frac{2 \times 3.44}{3 \times 1 \times 10^{-5}} = 229{,}000$$

From curve 6, Fig. 119, $f = 0.020$

$$h = f_2 \frac{L_2}{D_2}\frac{V_2^2}{2g} = 0.020 \times \frac{2000}{\frac{2}{3}} \times \frac{\overline{3.44}^2}{2g} = 11.0 \text{ ft.} \qquad \textit{Ans.}$$

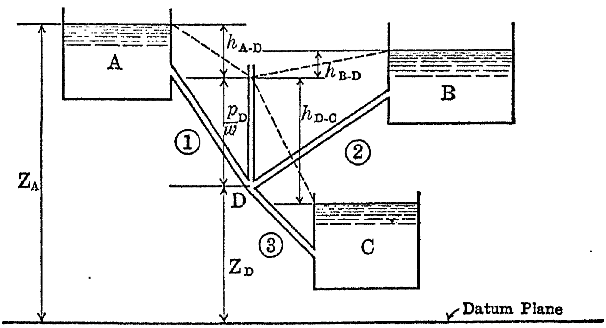


Fig. 135

98. Branching Pipes. — In water supply practice, it is sometimes necessary to analyze flow conditions in a system of branching pipes similar to that represented in Fig. 135. The elevations of the surface of the water

in each reservoir and the size, length and kind of pipe are known. It is necessary to determine the discharge and direction of flow in each pipe.

A solution could be obtained quite readily if the elevation of the hydraulic gradient at the juncture D were known. In this case, writing Bernoulli's equation between reservoir A and D, and neglecting minor losses, we would obtain

$$\frac{p_A}{w} + \frac{V_A^2}{2g} + Z_A = \frac{p_D}{w} + \frac{V_D^2}{2g} + Z_D + f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g}$$

If the pipes are relatively long, the velocity head is small in comparison to the head lost in friction and may be neglected. Then

$$0 + 0 + Z_A = \frac{p_D}{w} + Z_D + f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g}$$

and

$$Z_A - \left(\frac{p_D}{w} + Z_D\right) = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} = h_{A-D} \tag{a}$$

which is the difference in elevation between the surface of reservoir A and the hydraulic gradient at D. Similar expressions for the other reservoirs may be written and we have

$$h_{B-D} = f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} \tag{b}$$

$$h_{D-C} = f_3 \frac{L_3}{D_3} \frac{V_3^2}{2g} \tag{c}$$

Using the above equations, a discharge through each pipe can be determined by the methods previously explained. If the elevation of the gradient at D is above that of reservoir B, flow will take place into the reservoir and

$$Q_1 = Q_2 + Q_3 \tag{d}$$

When the gradient at D is below reservoir B

$$Q_1 + Q_2 = Q_3 \tag{e}$$

The elevation of the gradient at D is not originally known. The procedure is to assume an elevation for the gradient at D. The discharges are then computed and checked against Eq. (d) if the gradient was assumed above reservoir B, and against Eq. (e), if assumed below reservoir B. When Eq. (d) or Eq. (e) is satisfied, the problem is solved. It will be found easier to use reasonable assumed constant values of f in order to obtain an initial agreement of the equations. The approximate discharges

obtained in this manner may then be adjusted for new and more accurate values of the friction factors and consequently more accurate values for the discharges.

Illustrative Problem: The three reservoirs in Fig. 136 are connected by the pipes shown. Water with a kinematic viscosity of 1×10^{-5} sq. ft. per sec. is flowing. Find the quantity and direction of flow in each pipe.

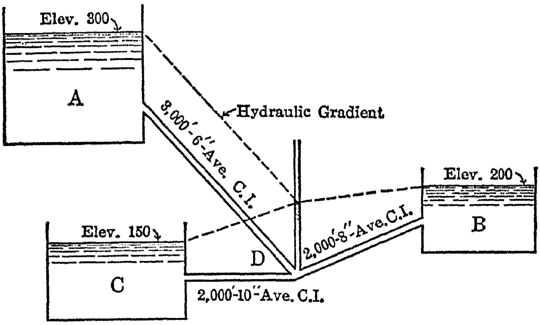


FIG. 136

Assume $f = 0.02$ for all pipes, and the elevation of the hydraulic gradient at D to be 225 ft.

$$h_{A-D} = 300 - 225 = 75 \text{ ft.}$$

$$h_{B-D} = 225 - 200 = 25 \text{ ft.}$$

$$h_{D-C} = 225 - 150 = 75 \text{ ft.}$$

For the 6 in. pipe

$$h_{A-D} = 75 = 0.02 \times \frac{3000}{\frac{1}{2}} \frac{V_1^2}{2g}$$

$$V_1 = \sqrt{\frac{75 \times 2g}{0.02 \times 6000}} \qquad 6.36 \text{ ft. per sec.;}$$

$$Q_1 = 0.196 \times 6.36 = 1.24 \text{ c.f.s.}$$

For the 8 in. pipe

$$h_{B-D} = 25 = 0.02 \times 2000 \frac{V_2^2}{2g}$$

$$V_2 \quad \sqrt{\frac{25 \times 2g}{0.02 \times 3000}} \qquad 5.19 \text{ ft. per sec.}$$

$$Q_2 = 0.349 \times 5.19 = 1.81 \text{ c.f.s.}$$

The hydraulic gradient assumption required that

$$Q_1 = Q_2 + Q_3$$

but

$$Q_2 > Q_1$$

Obviously the original assumption was incorrect.

Assume $f = 0.02$ and the elevation of the gradient at D to be 177 ft.

$$h_{A-D} = 300 - 177 = 123 \text{ ft.}$$

$$h_{B-D} = 200 - 177 = 23 \text{ ft.}$$

$$h_{C-D} = 177 - 150 = 27 \text{ ft.}$$

$$V_1 = \sqrt{\frac{123 \times 2g}{0.02 \times 6000}} \quad 8.12 \text{ ft. per sec.}$$

$$Q_1 \quad 0.196 \times 8.12 = 1.59 \text{ c.f.s.}$$

$$V_2 \quad \sqrt{\frac{23 \times 2g}{0.02 \times 3000}} \quad 4.97 \text{ ft. per sec.}$$

$$Q_2 = 0.349 \times 4.97 = 1.73 \text{ c.f.s.}$$

$$V_3 = \sqrt{\frac{27 \times 2g}{0.02 \times 2400}} = 6.02 \text{ ft. per sec.}$$

$$Q_3 = 0.545 \times 6.02 = 3.28 \text{ c.f.s.}$$

$$Q_1 + Q_2 = Q_3 \text{ very closely}$$

The friction factors should now be checked.

For the 6 in. pipe,

$$\mathbf{R} \quad \frac{6 \times 8.12}{12 \times 1 \times 10^{-5}} \quad 406{,}000$$

From curve 7, Fig. 119, $f_1 = 0.0197$.

For the 8 in. pipe,

$$\mathbf{R} = \frac{8 \times 4.97}{12 \times 1 \times 10^{-5}} = 331{,}000$$

From curve 7, Fig. 119, $f_2 = 0.0202$.

For the 10 in. pipe,

$$\mathbf{R} = \frac{10 \times 6.02}{12 \times 1 \times 10^{-5}} = 501{,}000$$

From curve 6, Fig. 119, $f_3 = 0.0183$.

Corrected values of discharge are $Q_1 = 1.61$ c.f.s., $Q_2 = 1.73$ c.f.s. and $Q_3 = 3.43$ c.f.s. These values check closer than the probable error in the values of f obtained from Fig. 119, and the problem is considered solved.

PROBLEMS

206. Two pipes, A and B, are connected in parallel. Pipe A consists of 2000 ft. of 12 in. average cast iron pipe; pipe B is 1000 ft. of 8 in. clean steel pipe. Find the discharge through the system when the difference of head between the points of juncture is 20 ft. Water is flowing with $\nu = 1 \times 10^{-5}$ sq. ft. per sec.

207. Two pipes, A and B, are connected in parallel. Pipe A is of best cast iron and is made up of 700 ft. of 12 in. pipe, 500 ft. of 10 in. pipe, and 900 ft. of 8 in. pipe.

Pipe B is 1000 ft. of 12 in. average cast iron. Find the discharge through the system when the difference in head between the points where the two pipes are connected is 30 ft. $\nu = 1 \times 10^{-5}$ sq. ft. per sec. *Ans.* $Q = 11.18$ c.f.s.

208. Find the discharge in each pipe, and the loss of head when 4 c.f.s. flow through the system of Prob. 206.

209. Find the discharge in each pipe and the loss of head when 3 c.f.s. flow through the system of Prob. 207.

210. Pipes 1, 2, and 3, having diameters of 10 in. and leading from reservoirs A, B, and C respectively, join at a common point. The lengths of the pipes are 5000 ft., 2000 ft., and 2000 ft., respectively. All pipes are average cast iron. Reservoir A is 200 ft. above C, and B is 100 ft. above C. Find the discharge and direction of flow in each pipe. Water is flowing with $\nu = 1.2 \times 10^{-5}$ sq. ft. per sec.

Ans. $Q_1 = 4.44$, $Q_2 = 2.07$, $Q_3 = 6.51$ c.f.s.

211. The pressure head at point A in Fig. 137 is 240 ft. All pipes are clean steel and the fluid is water at 50° F. Find the discharge in each pipe.

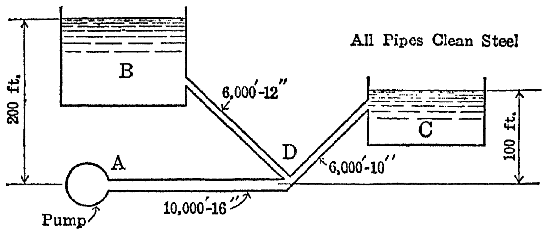


Fig. 137

99. Friction Loss in Flow of Compressible Fluids. — When a compressible fluid flows through a pipe, there is an expansion of the fluid along the pipe due to the pressure drop. The specific weight of the fluid decreases in the direction of flow. Since continuity of flow requires that the same weight of fluid pass any cross section in the same interval of time, there must be an increase in velocity accompanying the decrease in specific weight.

The fundamental equation for flow of compressible fluids including friction was derived in differential form in Art. 47, and is

$$vdp + \frac{VdV}{g} + dZ + dh_f = 0 \tag{55a}$$

In most problems, the term involving the elevation, Z, is small and is usually neglected. Thus Eq. (55a) may be written

$$vdp + \frac{VdV}{g} + dh_f = 0 \tag{a}$$

For a differential length, dL, the head loss from the Darcy equation is

$$dh_f = f\frac{dL}{D}\frac{V^2}{2g}$$

Substituting in (a)

$$vdp + \frac{VdV}{g} + f\frac{dL}{D}\frac{V^2}{2g} = 0 \tag{b}$$

Dividing by $V^2/2g$

$$\frac{2gvdp}{V^2} + \frac{2dV}{V} + f\frac{dL}{D} = 0 \tag{c}$$

Since

$$W = wAV, pv = RT, \quad \text{and} \quad v = \frac{1}{w}$$

$$V = \frac{W}{wA} = \frac{Wv}{A} = \frac{WRT}{pA}$$

$$\frac{W^2R^2T^2}{p^2A^2} \quad \text{and} \quad v = \frac{RT}{p}$$

Substituting for v and V^2 in (c), we obtain

$$\frac{2gA^2}{W^2RT}pdp + \frac{2dV}{V} + f\frac{dL}{D} = 0 \tag{d}$$

The equation can be integrated if f is constant along the pipe. This is true when the flow is isothermal. The value of f for a given pipe depends upon Reynolds number

$$\mathbf{R} = \frac{DV\rho}{\mu} = \frac{DVw}{\mu g}$$

Since, from continuity, wAV is a constant, wV is constant for a pipe of uniform diameter; μ is also a constant for a gas at constant temperature. Therefore $\mathbf{R}$ and f are both constant for isothermal conditions. Assuming isothermal conditions, Eq. (d) integrates as

$$2gA^2\left(\frac{p_2^2 - p_1^2}{2}\right) + 2\log_e\frac{V_2}{V_1} + f\frac{L}{D} = 0$$

or

$$p_1^2 - p_2^2 = \frac{W^2RT}{gA^2}\left(f\frac{L}{D} + 2\log_e\frac{V_2}{V_1}\right) \tag{151}$$

in which p_1 = pressure at (1) in pounds per square foot absolute,
p_2 = pressure at (2) in pounds per square foot absolute,
W = weight flow in pounds per second,
R = gas constant,
T = absolute temperature,
A = area of cross section in square feet,
L = length over which pressure drop occurs in feet,
D = diameter in feet,
V_1 and V_2 = velocities at (1) and (2) respectively in feet per second.

Ordinarily, in problems involving flow of compressible fluids in a pipe, the pressure p_1, the temperature and the weight flow are known. The pressure p_2 is required. Since V_2 depends upon p_2, Eq. (151) cannot be solved directly. However, the term involving the velocities is usually small compared to the friction term and may be neglected at first. An approximate value of p_2 may then be obtained. Since

$$w_1 A_1 V_1 = w_2 A_2 V_2$$

and

$$pv = RT$$

$$\frac{V_2}{V_1} = \frac{w_1}{w_2} = \frac{p_1}{p_2}$$

By substituting this approximate value of V_2/V_1 in Eq. (151), a more accurate determination of p_2 can be made.

It was stated early in this chapter that gases may be considered incompressible when the pressure changes are small. Under these conditions, the less complicated Darcy equation could be used. The error introduced by the use of the Darcy equation will now be considered.

Neglecting the last term in Eq. (151) and substituting $w_1 A_1 V_1$ for W

$$p_1^2 - p_2^2 = \frac{w_1^2 A^2 V_1^2 RT}{gA^2}\left(f\frac{L}{D}\right)$$

Factoring $(p_1^2 - p_2^2)$ and substituting p_1/w_1 for RT

$$(p_1 - p_2)(p_1 + p_2) = 2w_1 p_1 f \frac{L}{D}\frac{V_1^2}{2g}$$

and

$$h_f = \frac{(p_1 - p_2)}{w_1} = \left(\frac{2p_1}{p_1 + p_2}\right) f \frac{L}{D}\frac{V_1^2}{2g}$$

Thus, if p_2 is 10 per cent less than p_1,

$$\frac{2p_1}{p_1 + p_2} = \frac{2}{1 + 0.9} = 1.05$$

and the error introduced by using the Darcy equation is about 5 per cent. Since the friction factor may be in error by this amount, the simpler equation may be used for compressible flow when the drop in absolute pressure is less than 10 per cent of the initial absolute pressure.

When the flow is not isothermal, Eq. (151) cannot be used but the pipeline can be divided into short lengths. The loss for each short length can be found by using the Darcy equation and the total loss for the pipeline determined.

Illustrative Problem: Air flows isothermally through a 2 in. wrought iron pipe at the rate of 1 lb. per sec. The temperature is 80° F. The pressure at one point in the line is 80 lb. per sq. in. abs. The absolute viscosity of air at this temperature is 3.9×10^{-7} lb. sec. per sq. ft. $R = 53.3$. Find the pressure drop in 200 ft. of pipe.

$$w_1 = \frac{p_1}{RT_1} = \frac{80 \times 144}{53.3 \times (80 + 460)} = 0.399 \text{ lb. per cu. ft.}$$

$$V_1 = \frac{W}{w_1 A} = \frac{1}{0.399 \times 0.0218} = 115 \text{ ft. per sec.}$$

$$\mathbf{R} = \frac{DV\rho}{\mu} = \frac{2 \times 115 \times 0.399}{12 \times 32.2 \times 3.9 \times 10^{-7}} = 610{,}000$$

From curve 6, Fig. 119, $f = 0.0178$.
Substituting in Eq. (151)

$$(80 \times 144)^2 - p_2^2 = \frac{1^2 \times 53.3 \times (80 + 460)}{32.2 \times (0.0218)^2}\left[0.0178 \times \frac{200}{\frac{1}{6}} + 2 \log_e \frac{V_2}{V_1}\right]$$

Neglecting the last term for the time being

$$p_2^2 = (80 \times 144)^2 - \frac{1 \times 53.3 \times 540}{32.2 \times (0.0218)^2}[0.0178 \times 1200]$$

$$p_2 = \sqrt{133{,}300{,}000 - 40{,}200{,}000} = 9640 \text{ lb. per sq. ft.}$$

or 67.0 lb. per sq. in.

Since

$$\frac{V_2}{V_1} = \frac{p_1}{p_2}$$

$$2 \log_e \frac{V_2}{V_1} = 2 \log_e \frac{80}{67} = 0.36$$

This is quite small in comparison with the other term in the brackets and can generally be neglected.

The loss in pressure is, therefore,

$$p_1 - p_2 = 80 - 67 = 13 \text{ lb. per sq. in.} \qquad \textit{Ans.}$$

PROBLEMS

212. Air flows isothermally through a 6 in. clean steel pipe. The pressure at one point in the line is 100 lb. per sq. in. abs. The pressure 2000 ft. further downstream is 60 lb. per sq. in. abs. The temperature is 60° F. Find the weight flow. $R = 53.3$. The absolute viscosity of air at this temperature is 3.8×10^{-7} lb. sec. per sq. ft.

213. Methane flows at 70° F. through a 12 in. clean steel pipe 5 miles long. The discharge is 10 lb. per second. Find the final pressure if the initial pressure is 80 lb. per sq. in. abs. R for methane is 96.5; $\mu = 2.1 \times 10^{-7}$ lb. sec. per sq. ft.

214. Carbon dioxide flows through a 3 in. wrought iron pipe at a temperature of 60° F. The pressure at one point in the line is 50 lb. per sq. in. abs. and the velocity is 80 ft. per sec. Find the pressure and velocity 1000 ft. downstream. $R = 35.1$; $\mu = 3.0 \times 10^{-7}$ lb. sec. per sq. ft.

Ans. $p = 24.7$ lb. per sq. in., $V = 162$ ft. per sec.

215. Air flows isothermally through a 6 in. clean steel pipe at a temperature of 90° F. At one point in the line, the pressure is 150 lb. per sq. in. abs. What is the pressure 1000 ft. downstream from this point when the flow is 8 lb. per sec.? $R = 53.3$; $\mu = 4.0 \times 10^{-7}$ lb. sec. per sq. ft.

CHAPTER IX

UNIFORM FLOW IN OPEN CHANNELS

100. General Considerations. — An open channel is one such that the discharging medium is not surrounded on all sides by a solid boundary. From this, it is evident that there must be a free surface and that the fluid must be a liquid since a gas would expand and completely fill the channel. From this definition, an open channel would be any natural stream, artificial canal or flume, or a pipe flowing partly full. A sewer is an example of the open channel which is a pipe flowing partly full. Since the open channel flows under atmospheric pressure, it must depend upon the slope of the channel in order that the velocity may be maintained.

The relative roughness of open channels varies between wide limits as the surface changes from that of a smooth sewer pipe to that of a weed-choked canal in which intermittent flow occurs. Each different channel has its own roughness factor and will have a different velocity for the same shape, depth, and slope. For comparable conditions, the velocity will be greater for the smoother channels and, for purposes of computation, an estimate of the roughness of the channel must be made. The engineer is able only through experience to choose a reasonable value for this factor and even then, under extreme conditions, a choice of the roughness factor 25 per cent in error might be considered quite satisfactory. An error in the choice of this factor would be reflected directly in the computation of the discharge.

Uniform flow can exist for the flow in sewers providing the slope is just sufficient to maintain a constant velocity. It can also be approximated in straight sections of canals and flumes located some distance from a point of change in area or slope. The flow would not be uniform near these changes. Such uniformity does not exist for the natural streams. The cross-sectional area, shape, and alignment continually change from one section to another; but if a sufficiently short reach is considered, the flow may be considered uniform within the reach and the equations which follow can be applied.

Two terms appear in the open channel formulas which will now be defined. The *wetted perimeter* is the length of that portion of the periphery of the cross-sectional area of the liquid which is in contact with a solid boundary. In Fig. 138, the wetted perimeter would be the length of the broken line *ABCD*. The hydraulic radius is the ratio of the cross-sectional

area of the stream to the wetted perimeter. In Fig. 138, it would be the area $ABCD$ divided by the length of the broken line $ABCD$.

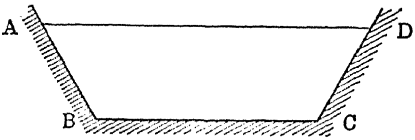


FIG. 138

101. The Chezy Formula. — One of the earliest formulas for determining the flow in open channels was proposed by Chezy in 1775. The Chezy formula may be derived as follows:

Referring to Fig. 139, it is evident that the resisting force under uniform flow conditions must just balance the component of the weight of the water in the direction of motion. The resisting force is equal to the product of the unit shear and the area of the solid boundary that is in contact with the moving water.

$$\tau(\text{w.p.})dl = wAdl \sin \theta$$

where τ = unit shear at the solid boundary,
w.p. = wetted perimeter,
dl = length of elemental volume,
w = weight of a cubic foot of the liquid,
A = cross-sectional area of the stream,
θ = angle of inclination of the liquid surface.

Solving for the unit shear, we obtain

$$\tau = \frac{wA \sin \theta}{\text{w.p.}}$$

It has been previously shown that τ varies approximately as the square of the mean velocity, V^2, for turbulent flow and since the inclination of the liquid surface is normally small, $\sin \theta$ can be replaced by $\tan \theta$ which is equal to the slope, S, of the surface. The ratio of the cross-sectional area to the wetted perimeter is the hydraulic radius, R. Making these substitutions, we obtain

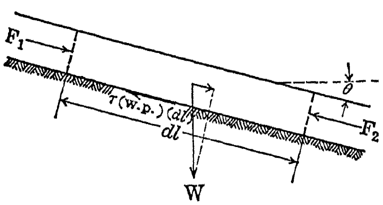


FIG. 139

$$V^2 \quad \frac{w}{k} RS$$

or

$$V = C\sqrt{RS} \qquad (152)$$

Equation (152) is known as the Chezy formula. While it does not resemble the Darcy formula for flow in pipes, it can be shown that it is closely related to it as follows:

Squaring Eq. (152) and remembering that $R = D/4$ and that $S = h_f/L$

$$V^2 = C^2RS$$

$$S = \frac{V^2}{C^2D/4} = \frac{h_f}{L}$$

or

$$h_f = \frac{LV^2}{C^2D/4}$$

By Darcy's equation

$$h_f = f\frac{L}{D}\frac{V^2}{2g}$$

Equating these two expressions

$$f = \frac{8g}{C^2}$$

or

$$C = \sqrt{\frac{8g}{f}} = \phi(\mathbf{R})$$

At the time that it was presented, Chezy felt that the value of C did not vary for a given stream. From the above discussion, one would conclude that C was a function of **R** and would not remain constant for varying flow conditions in a given stream. Referring to Kemler's curves, Fig. 119, we note that for extremely rough surfaces, or for high degrees of turbulence, f is actually independent of **R** and that the value of f really depends upon the relative roughness of the surface. With increasing stage in a given stream, the relative roughness decreases and the Chezy C does not remain constant for these changing conditions.

102. The Kutter Formula. — During the past century, many observations of the discharge in open channels have been made and a number of formulas have been proposed. Probably the most widely accepted of these was that proposed by Ganguillet and Kutter[1] in 1869 which is known as the Kutter formula. The purpose of this formula was to determine the value of C in the Chezy formula. The formula in English units and our notation follows:

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \dfrac{n}{\sqrt{R}}\left(41.65 + \dfrac{0.00281}{S}\right)} \qquad (153)$$

in which n is the roughness factor that is known as " Kutter's n."

[1] Ganguillet and Kutter, " Flow of Water in Rivers and Other Channels," translation by Herring and Trautwine, 2nd. Ed., John Wiley and Sons, Inc., 1901.

The equation is cumbersome and is not dimensionally correct. The fact that the equation is cumbersome does not offer serious objections since charts and tables have been prepared for the solution of the equation. With reference to the dimensions, it appears that the first two terms in the numerator are abstract, so in order for the numerator to be homogeneous, the n would have to be abstract. Similarly, the first term in the denominator is abstract and, in order for $n/R^{1/2}$ to be abstract, n would have the dimensions of $L^{1/2}$. Obviously, n cannot satisfy both of these requirements. It might be well to call the attention of the reader to the fact that C cannot be dimensionless, as reference to the Chezy formula will show. Analyzing the Chezy formula, it is evident that the dimensions of C must be $L^{1/2}T^{-1}$.

The terms which are dependent upon the slope were introduced so that the formula would satisfy the measurements made by Humphreys and Abbot on the Mississippi River. The discharges for these measurements were obtained by the float method and the slopes, the smallest of which were less than 0.02 ft. per mile, were obtained by the use of the engineer's level. It is now known that these discharge measurements were not accurate and, for this reason, the Kutter formula will not give reliable results for very low slopes.

Reliable results can be obtained by the use of the Kutter formula for normal slopes and for values of the hydraulic radius which lie between 1 and 10 feet. It has the advantage that it is widely used and that engineers are familiar with the numerical values of n. The formula is recognized and accepted in court proceedings.

103. The Manning Formula. — A more convenient formula was that proposed by Manning[1] in 1890. The formula is usually written

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{154}$$

where n has the same value as Kutter's n. This is equivalent to assigning the following value to C in Chezy's formula:

$$C = \frac{1.49}{n} R^{1/6}$$

Using the same values of n in the Manning and Kutter formulas, comparable values of discharge are obtained for the usual range in R and S; but, as already pointed out, the Kutter formula is not reliable for very flat slopes. For this reason, it is felt that the Manning formula is the most desirable equation and it is being more widely used. It is especially advantageous since the values of the roughness factor, n, are the same as

[1] Manning, Robert, "On the Flow of Water in Open Channels and Pipes," *Trans. Inst. of Civil Engrs. of Ireland*, V. 20.

those with which the engineering profession is familiar and the values of which have been determined for many different conditions.

Illustrative Problem: The discharge from a river was 715 c.f.s. when the cross-sectional area was 316 sq. ft., the hydraulic radius was 5.45 ft. and the slope was 0.0003. Find the value of the roughness factor n.

Using the Manning formula

$$V = \frac{1.49}{n} R^{2/3} S^{1/2}$$

we find

$$V = \frac{Q}{A} = \frac{715}{316} = 2.26 \text{ ft. per sec.}$$

$$n = \frac{1.49}{2.26} (5.45)^{2/3} (0.0003)^{1/2}$$

$$= 0.035$$

Ans.

Note: The side slopes and bottom of this river were of sandy clay loam and were very rough and uneven. The channel was fairly straight and there were few obstructions and little vegetation.

PROBLEMS

216. A river having a cross-sectional area of 760 sq. ft. and a hydraulic radius of 8.6 ft. discharged 3200 c.f.s. when the slope of the water surface was 0.000128. Find the roughness factor. *Ans.* $n = 0.0168$.

217. A dredged channel in loam soil had an area of 270 sq. ft. and a hydraulic radius of 4.2 ft. when the discharge was 615 c.f.s. Find the slope if the value of n was 0.028.

218. A river flowed through a crooked course in sandy clay loam soil. There were large variations in shape and there were many logs and other obstructions on the bottom and sides. Find the velocity in this stream when $R = 7.6$ ft., $S = 0.00149$ and $n = 0.14$.

104. Roughness Factors for Artificial and Natural Streams. — The roughness factors for different streams vary considerably for the different surfaces and generally vary somewhat for different stages in the same stream. The variation of n for different stages in the same stream is especially pronounced when the relative roughness of the sides of the stream above a low stage is considerably different from that of the bottom of the channel. Should the sides be relatively smooth, the value of n decreases for increasing stages, while if the sides are relatively rough, the value of n increases for increasing stages. This condition has been shown in the experiments reported by C. E. Ramser.[1]

The value of the roughness factor for a given type of surface does not remain constant with time. Let us consider a concrete flume which has

[1] Ramser, C. E., "Flow of Water in Drainage Channels," *U. S. Dept. of Agri., Tech. Bul., No.* 129.

a smooth surface. When new, the value of n for this flume would be approximately 0.013. If this flume is not covered in order to protect the interior from the sunlight, moss and other growth will form and attach itself to the sides of the flume. While this growth appears to be quite

TABLE VIII — VALUES OF n FOR USE IN MANNING'S FORMULA
From data from many sources

Nature of surface	n		
	Good	*Recommended value*	*Poor*
Neat cement surface	0.010	0.011	0.013
Wood stave pipe	.010	.011	.013
Smooth welded pipe	.009	.012	.013
Vitrified sewer pipe	.010	.012	.017
Plank flumes, planed	.010	.013	.014
Plank flumes, unplaned	.011	.015	.016
Cast iron pipe	.012	.013	.015
Commercial W.-I. pipe, black	.010	.013	.015
Concrete pipe or smooth flume	.011	.013	.018
Common clay drain tile	.011	.014	.017
Painted metal, wood stave, or concrete flumes under usual conditions		.014	
Brick in cement mortar	.012	.015	.017
Commercial W.-I. pipe, galvanized	.013	.015	.017
Any flume with heavy silt-and-moss accumulations or lime incrustations		.016	
Cement-rubble surface	.017	.023	.030
Corrugated metal	.022	.024	.030
Canals in earth, smooth	.014	.023	.025
in earth, dredged surface	.014	.028	.050
rough beds with roots and weeds	.025	.050	.120
rock cuts, smooth	.025	.032	.035
rock cuts, jagged	.035	.042	.050
Natural streams, smoothest	.018	.025	.028
fair with some vegetation	.025	.030	.050
irregular with vegetation	.030	.045	.070
much vegetation	.050	.110	.160

slippery, it will offer increased resistance to the flow and the value of n can increase to well over 0.020. Tubercles form on the interior of steel and cast iron pipe after they are in use and the capacity is reduced. The formation of the tubercles can be lessened by the application of a protective coating on the interior of the pipe. In much the same way, the carrying capacity of an open canal will decrease due to the growth of vegetation on the sides, and the carrying capacity will be less in the summer than during the winter due to the foliage on the growth.

From the above discussion, it is evident that no hard and fast rule can be prescribed for the choice of a proper coefficient. The values given in Table

VIII merely serve as a guide so that a reasonable choice of coefficient can be made. It is evident from the variation of the values in this table that the discharge of a natural stream which is in poor condition cannot be computed with a high degree of accuracy.

PROBLEMS

219. A smooth earth canal has a bottom width of 16 ft. and side slopes of 2 horizontal to 1 vertical. If the depth of the water is 5 ft. and the surface slope is 2 ft. per mile, find the velocity and the discharge.

220. A canal similar to that described in the preceding problem is designed to carry 325 c.f.s. with mean velocity of 2.5 ft. per sec. Find the required slope.

221. Water flows in a rectangular concrete flume, which is 12 ft. wide, at the rate of 200 c.f.s. Find the depth if $S = 0.0008$.

222. Find the discharge of a semicircular corrugated metal flume 8 ft. in diameter when flowing full with $S = 0.0016$.

223. Find the bottom width of dredged canal having side slopes of 2 horizontal to 1 vertical if it discharges 800 c.f.s. with a fall of 6 in. per mile. The depth is 8 ft.
Ans. $b = 42.6$ ft.

224. A river having a cross-sectional area of 1800 sq. ft. and a wetted perimeter of 315 ft. discharges 5000 c.f.s. with a value of $S = 0.0004$. Find the value of n.

225. A rectangular flume with a cement-rubble surface is 12 ft. wide and water flows with a 4 ft. depth. Find the discharge if the fall is 3 ft. per mile.

226. It is desired to increase the capacity of the flume in the preceding problem by lining it with cement mortar. Find the new discharge for the same depth of flow.

227. A rectangular planed plank flume, which is 10 ft. wide, carries 200 c.f.s. with a grade of 2 ft. per mile. The flume has a free-board of 9 inches. Find the inside dimensions of the flume.

228. A canal lined with concrete is to discharge 800 c.f.s. with a grade of 2.5 ft. per mile. The water surface is an average of 2 ft. below the ground level and the concrete lining is 6 in. thick. The side slopes are $1\frac{1}{2}$ horizontal to 1 vertical and the bottom width is 20 ft. The excavation costs 23 cents per cu. yd. and the concrete lining costs \$2.75 per sq. yd. Determine the dimensions of the canal and calculate its cost per mile of length.
Ans. Depth of excavation 7.47 ft., cost per mile = \$85,100.

105. Most Efficient Section. Consider the Manning equation for the discharge of a channel

$$Q = \frac{1.49}{n} A R^{2/3} S^{1/2}$$

For a given velocity in any one stream, A, n and S are constants, but the value of R can be made to vary by a change in the general shape and proportions of the cross section. We can see that with the other three properties remaining constant, the discharge will be a maximum when the value of R is a maximum. Since R is equal to the area divided by the wetted perimeter, R will be a maximum when the wetted perimeter is a minimum.

Now of all regular figures having areas of equal magnitude, the circle has the least wetted perimeter. The semicircle has one-half the area and one-half the wetted perimeter of the full circle, therefore, the flume whose section is the half circle would have the largest value of R, and correspondingly, a greater discharge than any other section having the same cross-sectional area. For this section,

$$R = \frac{A}{\text{w.p.}} = \frac{\pi r^2}{2(\pi r)} = \frac{r}{2} \tag{156}$$

or the hydraulic radius would be equal to one-half of the depth of flow. This section would be suitable for flumes made of galvanized iron, or of wood staves, but would not be satisfactory for flumes made of most engineering materials due to the difficulty of forming.

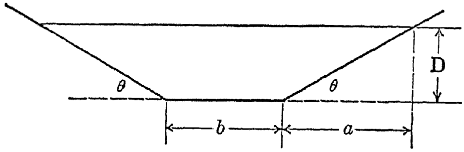


FIG. 140

A flume would normally be constructed with a rectangular section, while a canal in earth would have a trapezoidal cross section. The trapezoidal section would be used in the latter case due to the fact that the side slopes would have to be kept less than the angle of repose of the material. The best proportions for the trapezoidal section will now be considered:

Referring to Fig. 140, the area

$$A = bD + D^2 \cot \theta$$

from which

$$b = \frac{A - D^2 \cot \theta}{D} \tag{157}$$

The wetted perimeter

$$\begin{aligned} \text{w.p.} &= b + 2D \csc \theta \\ &= \frac{A - D^2 \cot \theta + 2D^2 \csc \theta}{D} \end{aligned}$$

from which

$$R = \frac{A}{\text{w.p.}} = \frac{AD}{A - D^2 \cot \theta + 2D^2 \csc \theta} \tag{158}$$

The maximum value of this expression is found by equating the first derivative to zero

$$\frac{dR}{dD} = \frac{A(A - D^2 \cot \theta + 2D^2 \csc \theta) - AD(4D \csc \theta - 2D \cot \theta)}{(A - D^2 \cot \theta + 2D^2 \csc \theta)^2} = 0$$

Collecting terms, we obtain

$$A + D^2 \cot \theta - 2D^2 \csc \theta \quad 0$$

or

$$A = D^2(2 \csc \theta - \cot \theta) \tag{159}$$

Substituting this value of A into Eq. (158), we obtain the value of R which will give the maximum rate of discharge in a given channel.

$$R = \frac{D^3 \,(2 \csc \theta - \cot \theta)}{D^2(4 \csc \theta - 2 \cot \theta)} = \frac{D}{2} \tag{160}$$

This value of R is identical to that which was obtained for the semicircular cross section. We will now substitute the value of A from Eq. (159) into Eq. (157) in order to determine the best bottom width.

$$b = \frac{2D^2 \csc \theta - 2D^2 \cot \theta}{D} = 2D(\csc \theta - \cot \theta) \tag{161}$$

From Eq. (161), the following relations between the bottom width and depth of flow have been obtained:

a/D	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4
b	$2D$	$1.24D$	$0.83D$	$0.61D$	$0.47D$	$0.32D$	$0.25D$

A semicircle can always be inscribed in a section of maximum efficiency, as is illustrated in Fig. 141 for the trapezoidal section having side slopes of 2 horizontal to 1 vertical.

The dimensions which we have found will produce the maximum discharge for a given area but they may not always be practicable, due to the fact that materials might be encountered at the greater depths which would unduly increase the cost of excavation. However, these dimensions should serve as a guide in making a proper choice of cross section and, in general, produce a canal having a minimum cross-sectional area and a channel which has the least wetted perimeter. Since both the cross-sectional area and the wetted perimeter are a minimum, both the amount of excavation and the area of lining material would be a minimum for this section and the cost per unit of length would be less than for any other shape.

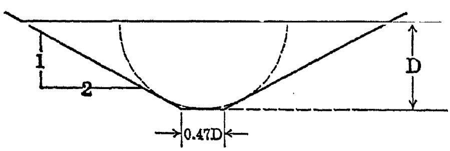


FIG. 141

PROBLEMS

229. A circular concrete sewer 6 ft. in diameter flows half full and has a grade of 5 ft. per mile. Compute the discharge.

230. Water flows in a rectangular planed plank flume at the rate of 70 c.f.s. with a grade of 4 ft. per mile. Determine the dimensions for the efficient section.

231. A canal in earth in good condition has side slopes of 3 horizontal to 1 vertical and carries 700 c.f.s. with a mean velocity of 3.5 ft. per sec. Determine the depth, bottom width and slope for the efficient section.

232. Determine the dimensions and the cost per mile of the canal described in Prob. 228 for the same discharge if the most efficient section were used.

233. A trapezoidal canal with side slopes of $1\frac{1}{2}$ horizontal to 1 vertical is to carry 400 c.f.s. with a velocity of 4.5 ft. per sec. The canal is lined with brick laid in cement mortar. Find the minimum amount of lining, in square feet per foot of length, and the slope of the canal. *Ans.* Area = 27.3 sq. ft.; S = 0.000428.

106. Flow in Rivers. — A continuous record of the stage and discharge of rivers is needed in the design of many engineering works. Among these are: (1) navigation structures, (2) sewage and industrial waste disposal plants, (3) design of drainage systems, (4) data for determining the quality and quantity of water available for processing purposes, and (5) the determination of the quantity of water available for power and irrigation purposes. It is necessary to know the low water discharge of a river before a steam plant can be built as large quantities of water are needed for the proper cooling of the condensers. Information on the flood flow of streams is needed in the design of levee systems and storage dams, and in the choice of a proper elevation for highway embankments and in the design of bridge openings. The information is also essential in the adjudication of water rights between persons and states.

Since many of our rivers flow through several different states and since the national welfare is so closely related to the knowledge of the runoff of our streams, the responsibility of securing the stream flow data has been largely assumed by our federal government, in cooperation with state and other agencies, and the results are published in the U. S. Dept. of The Interior Geological Survey Water Supply Papers.

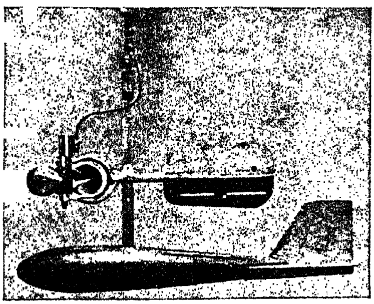


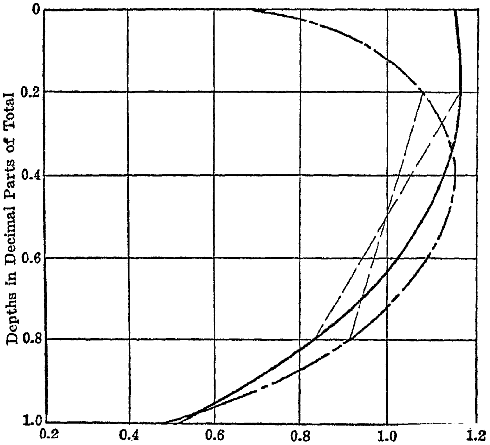
FIG. 142. The Price current meter with sounding weight. (Courtesy C. L. Muntz)

Prior to the measurement of the discharge of a stream, a metering section is chosen at a point along the stream where the bed and banks appear reasonably stable and accessible. In many cases, the measurement is made from a highway bridge but, at times, it may be necessary to suspend a cable across the stream and operate from a cable car. Measurements are sometimes made from a boat or by wading.

In making the measurement, the velocity of the water is measured by means of a current meter. The U. S. Geological Survey has adopted the modified small Price meter, Fig. 142, now called type-A-meter, as their

standard. The meter consists of a series of conical cups which are attached by means of a spider to a vertical shaft which is equipped with a commutator. As the flowing water causes the cups and shaft to rotate, the commutator makes and breaks an electrical circuit and a record of this is heard as clicks through the earphone which the operator wears. Readings are taken with the meter either mounted on a rod, or supported on a cable. If the meter is supported on a cable, it is held in place by means of a heavy lead weight which is directly below the meter.

After much study, it has been found that the velocity of the water in any vertical section of the stream follows the pattern indicated in Fig. 143. The pattern of this curve will vary considerably for different positions at



Velocity in Terms of Mean Velocity

——— Mean of 78 curves without ice cover

—·—·— Mean of 42 curves with ice cover

From U.S. Geological Survey—W.S. Paper No. 187.

FIG. 143. Velocity distribution in open channels.

one section in a river, and also for different wind conditions. With a stiff upstream wind, the open water curve would resemble the curve for ice cover; and with a downstream wind, the surface velocity would be relatively greater. Probably the most accurate value of the mean velocity in any vertical section could be obtained by measuring the velocities at a number of points in a vertical and then obtain the mean velocity from a graph upon which these values had been plotted, but such a procedure would be slow and costly. Excellent results can be obtained by observing the velocity at 0.2 and 0.8 of the depth. The mean of these values very closely approximates the true mean velocity. When the depth of the water is not sufficient so that the meter can be lowered to the 0.8 position, the

meter is placed at 0.6 of the depth and the value of the velocity which is observed at this depth is taken as the mean velocity in the vertical. The 0.6 depth method does not yield results having the high order of accuracy given by the 0.2 and the 0.8 depth method. At times, the depth is not sufficient for the 0.6 depth measurement, or the stream may contain much drift. At such sections, the surface velocity is observed and this value multiplied by a constant in order to obtain the mean velocity in the vertical. Surface observations yield values of mean velocity which are the least accurate, and this method is used only for sections at which it is not possible to use one of the other methods.

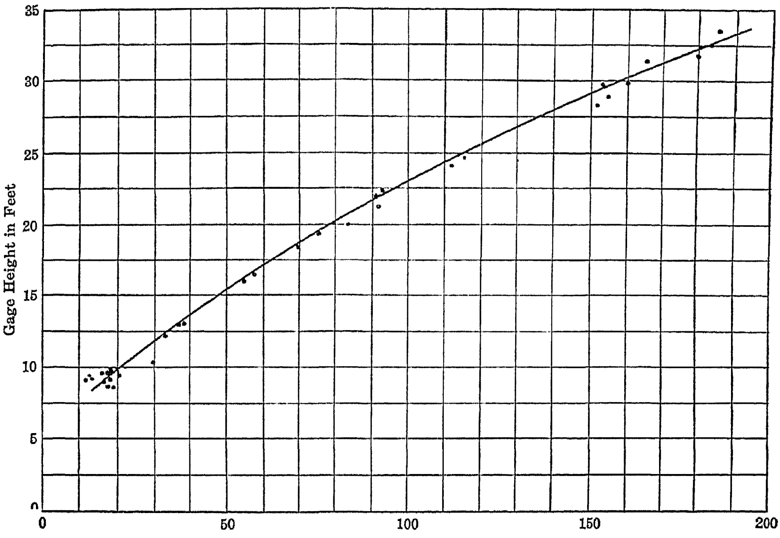


FIG. 144. Rating curve for Tennessee River at Chattanooga.

In order to determine the discharge of the stream, it is first divided into a number of sections which are preferably of equal width. The mean velocity in each of these vertical sections is determined by one of the methods outlined above and the area of the section, which is considered to be trapezoidal, is determined from soundings made with the meter just prior to the measurement of the velocity. The mean velocity in the section is taken as the mean of that in the two bounding verticals, and the discharge through the section is then the product of the area and the mean velocity. Corresponding discharges are determined for each of the other sections, and these are all added to obtain the discharge of the stream. After measurements of the discharge have been made for a number of different stages throughout the range in stage, a *rating curve*, such as is shown in Fig. 144 for the

Tennessee River[1] at Chattanooga, Tenn., is drawn. From a record of the stage of the stream, which may be obtained from observations on staff gages or from the chart obtained from a continuous water level recorder, the discharge may be read from the rating curve.

The staff gage, or the water level recorder, is placed a short distance upstream from the *control*. The control is a section in the stream which governs the stage needed for a given discharge. It may be very well defined, as the crest of a dam or a rock ledge in the stream; or less well defined if it is a uniform reach of channel, a rapids, or a chute in the stream. Excellent controls of the above type are quite often not available, and, in that case, a constriction in the width of the channel might serve, or the resistance to flow offered by the channel itself downstream from the gage would require a certain head at the gage in order to produce the existing discharge. This would be known as *channel control*. Controls of the latter types may not be well defined and positive throughout the entire range in stage and the discharge will be accompanied by a drowning action caused by some downstream condition. Should this occur, the effect will not be the same for different rates of change of stage for one gage height, and the discharge will be largely affected by the variable slopes. The rating curve will no longer be a single well defined curve and a considerable variation in discharge will be in evidence for a given gage height depending upon the slope of the energy gradient at the station. The Chattanooga station is a variable slope station and again referring to Fig. 144 it is evident that, while the curve which has been drawn may average the measured discharges quite well, any one measurement may miss the curve by a considerable amount, especially at the lower stages. In fact, the rating curve misses the measured discharges for the five lowest stages an average of about 25 per cent. Such poor agreement is not acceptable in practice for simple discharge determination and was caused by differences in the slope which varied from 0.0000329 to 0.0000854.

Such variations in slope can be caused by a sudden rise, or fall, in the stream; or by backwater effects from farther downstream. Two recorders are needed in order to properly determine the slope and discharge of the stream and this increases the annual cost of the operation of the station considerably. Knowing the slope and the rate of change of the slope, the Geological Survey has developed rather involved methods for making corrections for differences due to the changing slope, but a discussion of these methods is beyond the scope of the present work. Suffice it to say that the methods which have been developed will generally produce an agreement within 3 per cent.

[1] The complete data from which the points on Figs. 144 and 145 were computed were furnished by C. E. McCashin, Dist. Engr., U. S. Geological Survey, Chattanooga office, to whom the authors are exceedingly grateful.

A rather simple method of determining the discharge, which can be used whenever the rate of change of slope need not be considered, would be to compute the value of Manning's "n" and plot a curve of "n" against the hydraulic radius "R." Such computations have been made for the measurements plotted in Fig. 144 to produce the curve of Fig. 145.

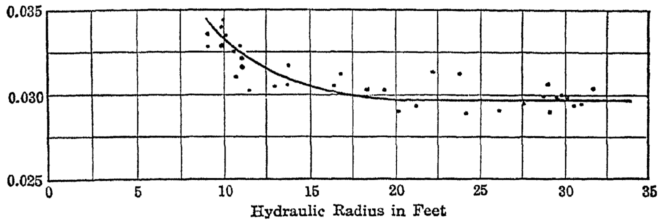


FIG. 145. Variation in Manning's "n" with changes in hydraulic radius for the Tennessee River at Chattanooga.

Examination of this curve will show that there is little variation between the experimental values of "n" and those obtained from the curve. The use of this type of curve for computing the discharges has reduced the apparent error of 25 per cent obtained from the rating curve for the 5 low readings to only 1.8 per cent. Such agreement is remarkable, but the need for two recorders still remains.

107. Irregular Sections. — The natural stream encountered in practice is not symmetrical in section nor uniform along its length. The stream normally has a steep bank on one side and a flat flood plain, subject to overflow, on the other such as is illustrated in Fig. 146. The open channel formula cannot be used in cases where there is a sharp break in the form of the wetted perimeter. This is illustrated by the following example. Let it be assumed that the entire area in Fig. 146 is 3000 sq. ft. and that the area of the portion abc is 2200 sq. ft. Let the wetted perimeter abc be 250 ft. and bde be 300 ft. Let $n = 0.033$ and $S = 0.00015$. Using Manning's formula and solving the problem in one step it follows that

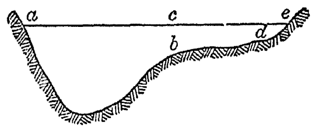


FIG. 146

$$Q \quad \frac{1.49}{0.033}(3000)\left(\frac{3000}{550}\right)^{2/3}(0.00015)^{1/2} = 5150 \text{ c.f.s.}$$

Now considering the portion abc and omitting the distance bc from the wetted perimeter since there is little frictional resistance on this portion,

we obtain

$$Q = \frac{1.49}{0.033}(2200)\left(\frac{2200}{250}\right)^{2/3}(0.00015)^{1/2} = 5190 \text{ c.f.s.}$$

It is evident that a portion of the area could not furnish a greater discharge than the entire area and that it would be necessary to break the area into two parts. In practice, the value of n would be considerably smaller for the main channel as compared to the value for the floodway.

PROBLEM

234. Assume that the cross section of a river and its floodway approximates two rectangles, one of which is 400 ft. wide and 18 ft. deep and the other 600 ft. wide and 5 ft. deep. Assume n is 0.028 for the main channel and 0.042 for the floodway. Find the discharge in one step using a mean value of n of 0.030. Find the discharge taking the two parts separately.

108. Transitions in Section. — While the general topic under discussion has been uniform flow, it would not be advisable to pass without some mention being made of transitions. Uniform flow does not exist near these points of transition and no attempt will be made to discuss the subject fully.

Points of transition occur at the entrance of a flume from a reservoir, a change in grade in a flume, at changes of shape or area of the flume, and at the outlet of the flume. The flow conditions at these points must receive careful consideration in order that the exact elevation of the water surface may be predicted. Some of the conditions which may arise will now be considered.

As water passes from a reservoir into a flume, there is a considerable drop in the water surface elevation, as shown in Fig. 147. A portion of the drop is due to the head lost at the entrance which is responsible for the energy gradient. The magnitude of this loss can be lessened by the use of easy curves at the entrance. The major part of the drop of the water surface is due to the building up of the velocity in the flume and is equal to $\alpha(V^2/2g)$. It is possible that the velocity at the entrance to the flume has reached the critical value. The reader will recall that the discharge over the broad-crested weir reached a maximum when the depth over the crest was $\frac{2}{3}H$. For this condition, the velocity head was one-half the depth of flow and the velocity for that condition was known as the critical velocity. Should the critical velocity be attained,

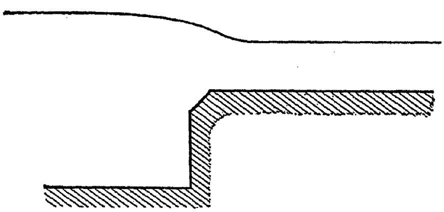


FIG. 147. Conditions at entrance to flume.

the flume intake would behave as a broad-crested weir and the discharge into the flume would be fully controlled by the entrance condition and an increase in the bottom slope of the flume would not cause an increased discharge. A greater flow could only be obtained by widening the flume or by lowering the elevation of the bottom of the flume at the entrance.

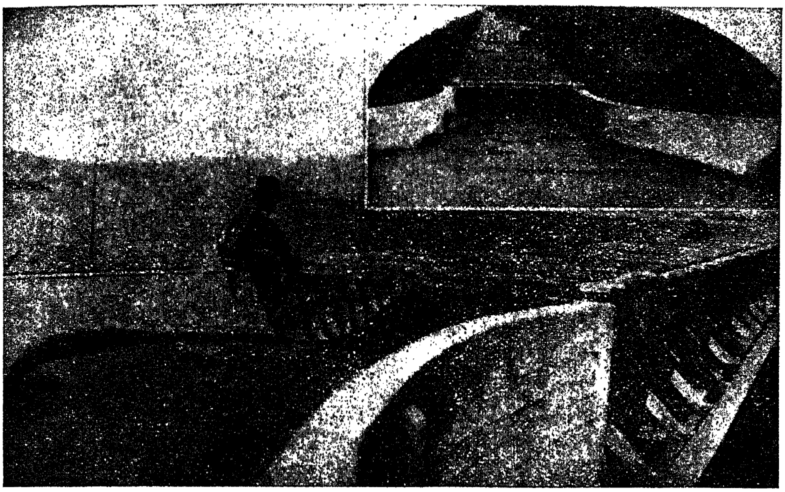


FIG. 148. Warped inlet McEachren flume, King Hill project Idaho. (From Tech. Bul. 393, U. S. Dept. of Agriculture by F. C. Scobey.)

Standing waves may easily be set up at these points of transition, as shown in Fig. 148 for an inlet on the King Hill irrigation project in Idaho. In spite of careful design, these inlets may develop rough water for discharges considerably below designed values and the ultimate discharge would be limited to some smaller value.

Conditions need not remain favorable in a uniform section of flume. This is illustrated in Fig. 149. The fall was excessive at the inlet to this section and a velocity in excess of the critical velocity was attained. The slope of the flume was not sufficient to maintain the high velocity and the high standing waves resulted. The phenomenon in the background where the depth of flow suddenly became greater and then remained at that value is known as the *hydraulic jump*. A large amount of energy is dissipated within the hydraulic jump and its presence within a flume is generally objectional. However, it is sometimes used to dissipate energy at the toe of an overflow type dam.

Conditions which cannot be easily predicted may also occur at transitions

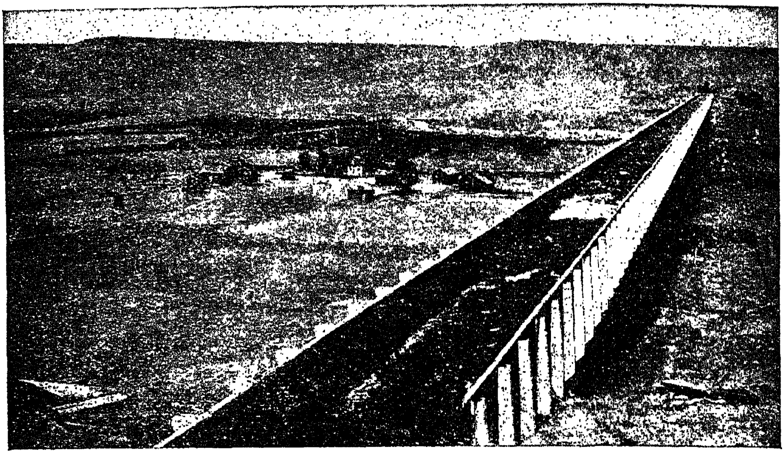


FIG. 149. Steep slopes at entrance to flumes fail to increase discharge. (Courtesy F. C. Scobey)

from one shape flume to another shape. The Bureau of Reclamation[1] constructed two transitions which were quite similar. One changed from a 5 ft.-6 in. circle to a 5 ft.-6 in. square while the other changed from a 6 ft.-0 in. circle to a 6 ft.-0 in. $\times$ 5 ft.-9 in. rectangle. The alignment in both cases was comparable and the overall length was 26 ft. in the first case and 27 ft. in the second. There was no reason from the plans to believe that the two would behave differently. When these transitions were placed in operation, it was found that the second one was quite satisfactory, but the first was so inefficient as to make it necessary to reconstruct it. May it be emphasized that this condition was experienced by the U. S. Bureau of Reclamation under whose direction many hydraulic structures have been designed.

The outlet structure, such as from a flume to a reservoir, is also of importance. Should the design of this structure be poor, an uneven velocity distribution may exist and there may be a tendency for scour to occur at the downstream edge of the structure. The scour at this point could cause undermining and failure of the structure.

The attempt has been made to emphasize that certain phenomena in hydraulic structures cannot be foreseen even by the experienced hydraulic engineer. For this reason, it is advisable that scale models of all important structures be constructed and tested prior to construction of the structure in order that the designers may know that the operation will be satisfactory, or so that changes can be made if these are desirable.

[1] Hinds, Julian, "The Hydraulic Design of Flume and Siphon Transitions," *T.A.S.C.E.*, V. 92, p. 1430, 1928.

CHAPTER X

DYNAMIC ACTION OF FLUIDS

109. Vector Quantities. — Their Addition and Subtraction. — The dynamic action of fluids cannot be discussed without frequent reference to vectors and vector quantities. In order to review somewhat and to familiarize the reader with the nomenclature that will be used, certain facts concerning vectors will be presented at this time.

A *vector quantity* is any quantity having magnitude and a sense of direction. Examples of vector quantities are forces, accelerations, velocities, momentums, etc. Any vector quantity may be represented by a vector, such as A or B in Fig. 150. The length of the vector is proportional to the magnitude of the quantity and its direction is the same as that of the quantity which it represents. The vector quantity normally has a position. For example, we cannot conceive of a force that does not act at some definite position, and the effect which it produces upon some body is dependent upon its position. We do not have quite the same feeling with reference to velocities and accelerations, largely due to the fact that we often think in terms of rectilinear motion of a rigid body where all points have the same velocity or acceleration. Position may be important for these quantities as, for example, the total acceleration of some point on a rotating disc. In this case, the magnitude of both components is directly dependent upon the position of the point in question.

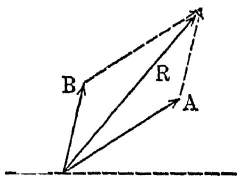


FIG. 150

The sum of two vector quantities is found by completing the parallelogram, as shown in Fig. 150, and drawing the diagonal R which is the resultant, or sum, of the two vectors A and B. The following representation will be used to indicate the vector addition:

$$A \mathbin{+\!\!\!\!\rightarrow} B = R \tag{162}$$

and will be read: B added vectorially to A equals R.

It is not essential to actually complete the parallelogram. It would suffice to draw vector B from the head end of vector A and then to draw R from the tail end of vector A to the head end of vector B. This would be known as the triangle method and it gives the correct result for the magnitude and direction of the sum R.

Vector subtraction will be indicated by the following convention:

$$A \rightarrow B = D \tag{163}$$

It is important to realize that $A \rightarrow B$ does not equal $B \rightarrow A$. In the parallelogram of Fig. 150, the diagonal which has not been drawn would represent the vector difference of the two quantities A and B. With the arrow at the upper end, we would have $B \rightarrow A$, while with the arrow at the lower end, we would have $A \rightarrow B$. It is evident that the two are numerically the same, but they act in just the opposite directions.

There is little chance of mistaking the direction of the vector difference if the operation is performed by means of addition and the triangle method, keeping in mind that

$$A \rightarrow B = A \mathrel{+\!\!\!\rightarrow} (-B) \tag{164}$$

It is then only necessary to consider vector B reversed in direction and to draw this reversed vector from the head end of vector A. The difference D is then drawn from the tail end of A to the head end of the reversed vector B. Should the difference $B \rightarrow A$ be desired, it would be obtained in the same way, using $B \mathrel{+\!\!\!\rightarrow} (-A)$. In Fig. 151, the vector differences $A \rightarrow B$ and $B \rightarrow A$ have been found.

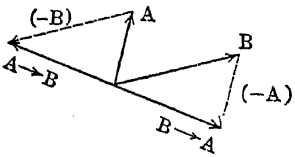


FIG. 151

In the study of the action of fluids on vanes, both absolute and relative velocities appear and they will be considered in a general way at this time. The *absolute velocity* of a body is its velocity referred to the earth, while the *relative velocity* of a body is its velocity with reference to another body which is itself moving with reference to the earth. This can be illustrated by considering a swimmer in a river. Let it be assumed that he is swimming perpendicular to the current and that his velocity with reference to the water is 2 mi. per hr. while the water has a velocity of 3 mi. per hr. Let us use the subscript M to represent the swimmer and W to represent the water. The 2 mi. per hr. is the relative velocity of the man with reference to the water and the 3 mi. per hr. is the absolute velocity of the water. If the absolute velocity of the man is now desired, which is diagonally downstream, it can be found by making use of the general equation

$$V_M = V_{M/W} \mathrel{+\!\!\!\rightarrow} V_W \tag{165}$$

In words, this equation would be stated as: the absolute velocity of the man equals the velocity of the man with reference to the water plus the absolute velocity of the water.

The terms in Eq. (165) can be transposed should one of the two quantities on the right be desired, thus

$$V_{M/W} = V_M \rightarrow V_W$$

It is probably always desirable to use Eq. (165) and then obtain the expression for the desired quantity by transposition.

Illustrative Problem: Water flows outwardly from a wheel having radial vanes with a velocity with reference to the wheel of 15 ft. per sec. The wheel is 4 ft. in diameter and is turning at the rate of 240 r.p.m. Find the absolute velocity of the water as it leaves the wheel. The conditions are as shown in Fig. 152.

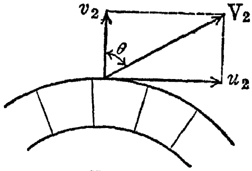


Fig. 152

Let V_1 and V_2 represent the absolute velocity of the water at the entrance and the exit to the vane, v_1, and v_2 the relative velocity of the water with respect to the vanes at the entrance and the exit, and u_1 and u_2 the absolute velocity of the vanes at the two points.

$$u_2 = \omega r = 2\pi\left(\frac{240}{60}\right)(2) = 50.2 \text{ ft. per sec.}$$

The absolute velocity of the water may now be found by the use of the equation

$$V_2 = v_2 \mathbin{+\!\!\rightarrow} u_2$$

$$V_2 = \sqrt{(15)^2 + (50.2)^2}$$

$$= 52.4 \text{ ft. per sec.}$$

$$\theta = \cos^{-1}\frac{15}{52.4} = 73^\circ\, 23'$$

110. Conditions Which the Vanes Should Satisfy. — As fluid enters a machine, whether it be a centrifugal pump, fan, or a hydraulic or steam turbine, the velocity V_1 acts in some rather fixed direction. If the fluid with this velocity moves into the machine with no tendency towards impact on either the front or rear face of the vane, the condition of *shockless entrance* is said to exist. Should the magnitude and direction of the velocity of the entering fluid not be correct for shockless entrance, impact will occur on the face of the vane, eddies will be set up and the efficiency of the machine will be lowered.

Assuming a fixed direction and magnitude for the initial velocity of the fluid and assuming a given speed for the vane, the required initial direction for the vane can be found. The assumed conditions, however, do not remain fixed. Consider a steam turbine, the velocity of the steam is dependent upon the initial pressure of the steam, and this pressure varies with load and firing conditions. The same is true for the hydraulic turbine. The velocity of the water entering the turbine depends upon the *head* on

the plant. This head is considerably different during high and low stages of the river and therefore the magnitude of the initial velocity of the water changes considerably. In the modern hydraulic turbine, the direction of the initial velocity of the water also changes for different load conditions. This is due to the fact that governing is accomplished by changing the gate opening and the wickets in the gates also serve as the vanes that direct the water into the turbine.

The direction and magnitude of the velocity of the flowing medium can be changed within the machine by the use of smooth curves on the moving vanes. Sudden changes should be avoided. In general, the curvature of the vanes should be such as to produce a desirable outlet velocity both with respect to magnitude and direction.

A given machine will not operate over an extended range of conditions due to loss of efficiency which accompanies the shock. It is for this reason that pumps and turbines are designed and built for particular operating conditions. For the hydraulic turbine, these conditions would include the head, power, and speed requirements. Of course, some machines are inherently more efficient than others, or are more adaptable to a wide range of conditions, but this does not change the fact that all machines have a most efficient operating condition. The flow requirements are best met by the machine over the high efficiency portion of the operating curve.

111. Force Exerted on a Stationary Vane. — As a fluid is deflected by a vane, its velocity is changed and a force is required to produce the change. Problems of this type can be solved with either the force-acceleration equation or with the impulse-momentum equation. The impulse-momentum equation will be used in this text.

Given the fundamental equation

$$Ft = M\Delta V \tag{166}$$

we have for a fluid

$$M = \frac{wQt}{g}$$

Upon substitution in Eq. (166), it follows that

$$F = \frac{wQ}{g}\,\Delta V = \frac{W}{g}\Delta V \tag{167}$$

in which F is the force in pounds required to produce the change in velocity ΔV, and acts in the same direction as ΔV which is expressed in terms of feet per second; w is the weight of a cubic foot of fluid in pounds; Q is the discharge in cubic feet per second; g is the acceleration due to gravity in feet per second per second; and W is the weight discharge in pounds per second.

The change in velocity, ΔV, is a vector quantity which is obtained by use of the equation

$$\Delta V = V_2 \rightarrow V_1 \tag{168}$$

Let the condition be as illustrated in Fig. 153*a*. It is normally assumed that the vane is frictionless, but this assumption is not essential. For a vane of this sort, the reduction in magnitude of the velocity relative to the

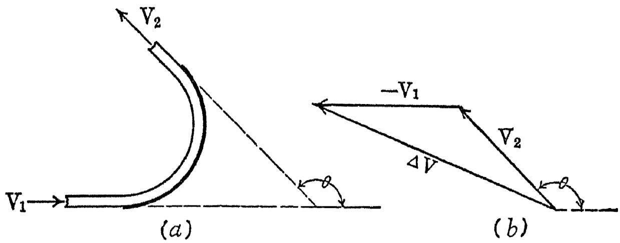


FIG. 153

vane would not be great due to the fact that the resisting force, which is tangent to the surface of the vane and which occurs because of the viscosity of the fluid, acts on the individual particles of fluid during only the short interval of time during which the fluid is on the vane.

From Fig. 153*b*, it is evident that

$$\Delta V_x = V_2 \cos\theta - V_1$$

and

$$\Delta V_y = V_2 \sin\theta$$

from which

$$F_x = \frac{W}{g}(V_2 \cos\theta - V_1) \tag{169}$$

and

$$F_y = \frac{W}{g}(V_2 \sin\theta) \tag{170}$$

and

$$F = \sqrt{(F_x)^2 + (F_y)^2}$$

The force exerted by the fluid is equal in magnitude to F, but acts in the opposite direction.

PROBLEMS

235. A 3 in. jet of water having a velocity of 120 ft. per sec. strikes a stationary curved vane and is deflected through an angle of 60°. Find the x- and y-components of the force which is exerted on the vane.

236. Given the conditions in Prob. 235 except that the final velocity is reduced to 108 ft. per sec. by friction on the vane. Find the components of the force which is exerted on the vane.

Ans. $F_x = 751$ lb.; $F_y = 1064$ lb.

237. A jet of water 2 in. in diameter is directed against a flat plate held normal to the stream's axis. Find the force exerted by the jet on the plate when the velocity of the jet is 115 ft. per sec.

238. A jet of water 1 in. in diameter exerts a force of 200 lb. against a flat plate which is held normal to the jet. Find the discharge.

Ans. 0.75 c.f.s.

239. Air at atmospheric pressure and temperature of 75° F. strikes a stationary curved vane with a velocity of 180 ft. per sec. and is turned through an angle of 180°. The jet of air is 3 in. in diameter. Find the force exerted on the vane.

Ans. $F = 7.31$ lb.

112. Force Exerted on Pipe Bends, Reducing Bends, and Reducers. — As a fluid flows around a pipe bend, forces are exerted on the bend due to the change in momentum and also due to the pressure within the bend.

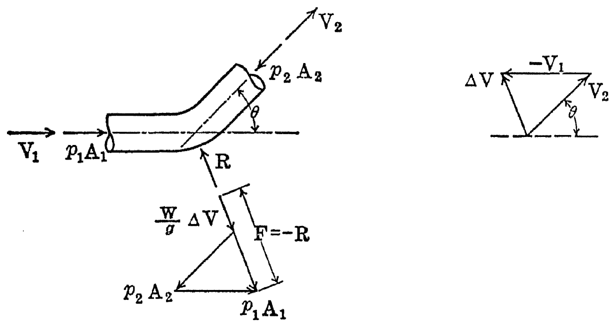


FIG. 154

The bend is illustrated in Fig. 154. The initial and final velocities are equal and p_1 does not differ greatly from p_2. The bend must resist the resultant, F, of the three forces making it essential for the bend to be anchored if stress is to be eliminated in the connecting flanges.

The method of solution can best be explained by the following example:

Illustrative Problem: Water flows around a 135° bend in a 12 in. pipeline at the rate of 9 c.f.s. The pressure at the bend is 50 lb. per sq. in. Find the magnitude

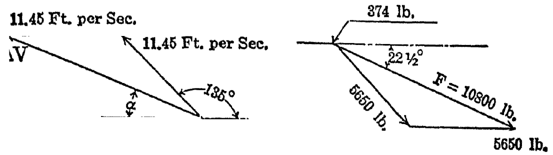


FIG. 155

and direction of the force exerted on the bend. The vector diagrams for the velocities and forces are given in Fig. 155.

$$V = \frac{9}{0.7854} = 11.45 \text{ ft. per sec.}$$

$$\Delta V_x = 11.45(1 + 0.707) = 19.55 \text{ ft. per sec.}$$

$$\Delta V_y = 11.45(0.707) = 8.10 \text{ ft. per sec.}$$

$$\frac{W}{g}\Delta V_x = \frac{62.4 \times 9}{g} \times 19.55 = 346 \text{ lb.}$$

$$\frac{W}{g}\Delta V_y = \frac{62.4 \times 9}{g} \times 8.10 = 141 \text{ lb.}$$

$$p_1A_1 = p_2A_2 = 50 \times 113 = 5650 \text{ lb.}$$

$$p_2A_{2x} = p_2A_{2y} = 5650 \times 0.707 = 3990 \text{ lb.}$$

$$F_x = 346 + 3990 + 5650 = 9986 \text{ lb.}$$

$$F_y = 141 + 3990 = 4131 \text{ lb.}$$

$$F = \sqrt{(9986)^2 + (4131)^2} = 10{,}800 \text{ lb.} \qquad \textit{Ans.}$$

$$\theta = \sin^{-1} \frac{4131}{10{,}800} = 22° \, 30' \qquad \textit{Ans.}$$

The conditions in the reducing bend are comparable to those in the bend of uniform diameter except that the pressure and velocity at the inlet and outlet are no longer the same. These values are computed by means of the equation of continuity and the Bernoulli equation. The vector diagrams are comparable to those appearing in Figs. 154 and 155.

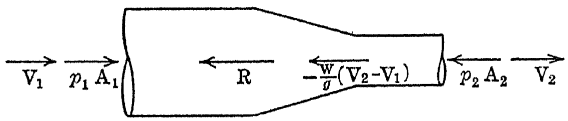


FIG. 156

A reducer, with the accompanying forces, is illustrated in Fig. 156.

The tensile force, R, in the coupling is given by the expression

$$R = p_1A_1 - p_2A_2 - \frac{W}{g}(V_2 - V_1) \qquad (171)$$

A force similar to that given in Eq. (171) would exist in the threads connecting a nozzle to a hose or pipeline, except that in this case, p_2 would normally be equal to zero.

PROBLEMS

240. Water under a pressure of 85 lb. per sq. in. flows around a 24 in. 90° bend at the rate of 40 c.f.s. Find the magnitude of the force exerted on the bend.
Ans. $F = 55{,}400$ lb.

241. Find the cubic feet of concrete needed to anchor the bend of Prob. 240 if the coefficient of friction is 0.4. Concrete weighs 150 lb. per cu. ft.

242. Gasoline, S.G. = 0.72, flows around a 90° bend in a 4 in. line under a pressure of 30 lb. per sq. in. and exerts a force of 550 lb. Find the velocity and discharge in the line.

243. A 135° reducing bend changes from a 12 in. to a 6 in. diameter. The initial pressure is 100 lb. per sq. in. when water is flowing at the rate of 4 c.f.s. Find the magnitude and direction of the force on the bend.

244. A 4 ft. pipe gradually reduces to 2.5 ft. in diameter. Water flows through the reducer with an initial velocity of 6 ft. per sec. and an initial pressure of 50 lb. per sq. in. The loss of head in the reducer is $0.1(V_2^2/2g)$. Find the axial force developed in the pipeline due to the reducer.

245. A 3 in. nozzle is attached to a 10 in. pipe by means of flange bolts. Find the tensile force in the bolts when the pressure at the base of the nozzle is 80 lb. per sq. in. $C_d = C_v = 0.98$. *Ans.* $F = 5290$ lb.

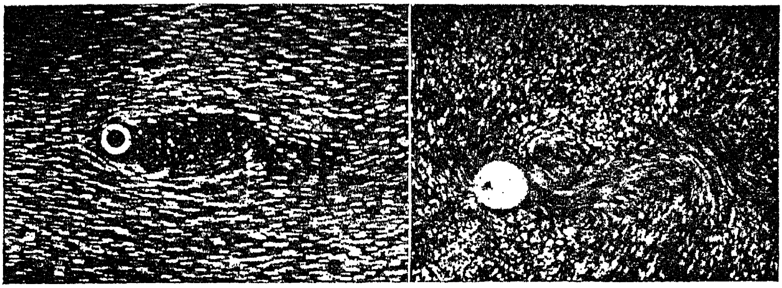
113. Dynamic Force on Cylinders and Spheres. — A knowledge of the force exerted by fluids on cylinders and spheres is important to engineers since these shapes are used from time to time. Examples are the wind resistance on spherical and cylindrical tanks, wind resistance on stacks, flagpoles or transmission lines, or the resistance offered by a river on a submerged pipeline. In the fields of aeronautics and automotive engineering, the resistance offered by other shapes are of tremendous importance, but these will not be considered. A similar discussion to that given here would apply to them, but the value of the coefficients would depend upon the degree of streamlining and the interference drag which is primarily set up at points of junction.

As a fluid flows past a spherical body, the fluid in the boundary layer near the sphere will be in laminar motion for low values of Reynolds number; but if the velocity becomes sufficiently high, motion in the boundary layer becomes turbulent. In either case, the fluid does not maintain contact with the surface of the sphere on the back side, but separates. From the point of separation, the conditions are very turbulent and pronounced eddies are formed. The eddy system is known as the *wake*, and the pres-

sure within it is considerably less than the static pressure in the undisturbed fluid. The major portion of the resistance, or *drag*, is due to the decreased pressure within the wake. The larger the transverse dimensions of the wake, the greater the drag coefficient. As the flow changes from laminar to turbulent in the boundary layer, the point of separation suddenly moves farther back on the body and the size of the wake is reduced. This change produces a sharp drop in the coefficient curve for a value of about 500,000 for the Reynolds number. The coefficient would probably remain about constant for the turbulent condition.

For low values of Reynolds number, say below a value of unity, the streamlines do not separate and the wake is not formed. For these low values, Stoke's law, which was derived from hydrodynamical reasoning, applies with little error. The resistance for the low values is caused by the viscous drag of the fluid on the surface.

The condition of a cylinder moving through still water[1] is shown in Figs. 157*a* and *b*. The Reynolds number for Fig. 157*a* was 72 and that



(*a*) (*b*)

Fig. 157. High and low velocity flow past cylinders.

for Fig. 157*b* was 1608. The separation and the wake are clearly evident in both cases, but the vortex motion in the wake is much more violent for the higher Reynolds number.

The same general conditions apply for the flow past the cylinder as for the sphere. If the cylinder is long, the flow approximates two-dimensional flow, while it would be three-dimensional for the sphere or for the short cylinder. The flow conditions for a short cylinder are more complex than those for a long cylinder. There is a flow over the end of the cylinder from the high pressure region on the front to the low pressure region within the

[1] The authors are indebted to E. E. Maser, Instructor in Aeronautical Engineering and to W. B. Stephenson, former student, Louisiana State University, for the loan of equipment and suggestions which were used in obtaining these photographs.

wake. This flow increases the pressure within the wake and decreases the drag which is exerted on the cylinder. The drag is reduced about 50 per cent for a length-diameter ratio of unity, about 30 per cent for a ratio of 10 and about 17 per cent for a ratio of 40.

The drag exerted on either the sphere or cylinder would be given by the general equation

$$F = C_d A \rho \frac{V^2}{2} \tag{172}$$

where F is the force in pounds, C_d is the coefficient of drag, A is the projection of the cross-sectional area in square feet normal to the direction of the velocity, ρ is the density of the fluid in slugs per cubic foot, while V is

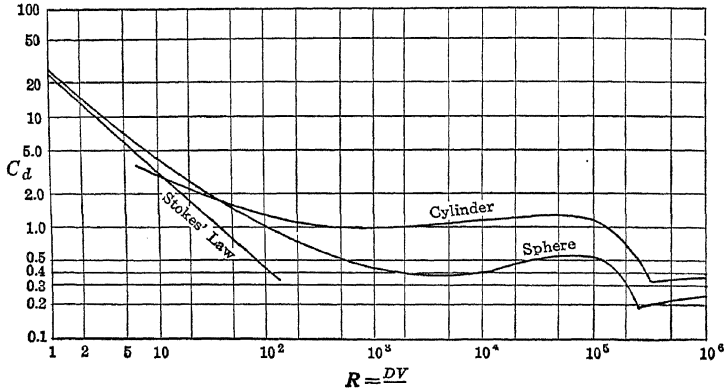


FIG. 158. The drag coefficient for cylinders and spheres.

the velocity of the fluid relative to the body in feet per second. The values of C_d for the sphere and cylinder are given in Fig. 158. The value of C_d given by Stoke's Law for the sphere is added for comparative purposes.

The magnitude of the critical Reynolds number, which occurs between about 200,000 and 500,000 and at which the value of C_d drops, is dependent upon the turbulence in the fluid stream which approaches the object. Keeping in mind that the drop in C_d occurs when the flow within the boundary layer changes from laminar to turbulent, it is obvious that the drop would occur for lower values of Reynolds number for the more turbulent conditions within the approaching stream. In fact, the value of this critical Reynolds number is used to define the turbulence of the approaching stream.

PROBLEMS

246. A spherical tank is 30 ft. in diameter. Find the drag on this tank when the wind velocity is 40 mi. per hr. Use $T = 50°$ F.

247. A pipeline, having a 12 in. outside diameter, extends across a river whose width is 1500 ft. Assuming that the water, whose mean velocity is 4.5 ft. per sec., has free access to the pipe, find the force exerted on the pipe. $T = 45°$ F. (Such a pipeline would actually rest on the bottom of the river and would be more or less buried. The water would not have free access to it.)

248. A flagpole is 150 ft. high. Considering it to have a mean diameter of 6 in., find the overturning moment caused by a 100 mi. per hr. wind when $T = 70°$ F. Should the pole be guyed? *Ans.* $M = 47{,}700$ ft. lb.

249. A stack is 150 ft. high and 6 ft. in diameter. Neglecting the end effect, find the overturning moment exerted on the stack by a 50 mi. per hr. wind when $T = 60°$ F.

114. Force Exerted on Moving Vanes. — Whenever a moving fluid is deflected by a curved vane, the vane exerts a force on the fluid; however, no work is done unless the vane is in motion. Two conditions exist: the fluid strikes an isolated vane, or it strikes a series of vanes which are mounted on a wheel. These conditions will be discussed.

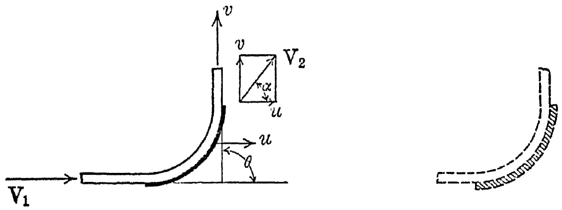


FIG. 159

(*a*) *Impact on a single vane; friction neglected.* The case of impact on a single vane is illustrated in Fig. 159 with an initial and a later position of the vane shown. The chief difference between this case and that for a stationary vane is in the weight of fluid striking the vane in a given period of time. In the former case, the full amount W lb. per sec. was deflected. Such is now no longer the case since the particles in the jet are overtaking the vane with a velocity of $V_1 - u = v$ ft. per sec. Letting W' be the weight discharge actually passing over the surface of the vane, then

$$W' = W\frac{V_1 - u}{V_1} \qquad (173)$$

It is now necessary to determine the final velocity, V_2, on the basis of the relative and absolute velocities. Thus $v = V_1 - u$ and, assuming no loss of velocity on the vane, $V_2 = u \nrightarrow v$. This relationship is indicated in Fig. 159.

The force in the direction of the jet is

$$F_x = \frac{W'}{g}(V_2 \cos \alpha - V_1)$$

but

$$V_2 \cos \alpha = u + v \cos \theta$$

$$= u + (V_1 - u) \cos \theta$$

$$F_x = \frac{W'}{g}(u + (V_1 - u) \cos \theta - V_1)$$

$$= \frac{W'}{g}(V_1 - u)(\cos \theta - 1)$$

$$= \frac{W}{g}\frac{(V_1 - u)^2}{V_1}(\cos \theta - 1) \tag{174}$$

$$F_y = \frac{W'}{g}(V_1 - u) \sin \theta$$

$$= \frac{W}{g}\frac{(V_1 - u)^2}{V_1} \sin \theta \tag{175}$$

(*b*) *Impact on a series of vanes; friction neglected.* Instead of having one vane which moves farther and farther from the nozzle, a series of vanes may be attached to a wheel. If the radius of the wheel is large in comparison to the size of the vane and jet, rectilinear motion can be approximated. The only way in which this case differs from that of the single vane is in the weight flow actually coming in contact with the vanes. There is now a constant distance between the nozzle and the series of vanes and the entire weight W is effective in producing the force. The expressions for the components of the forces are

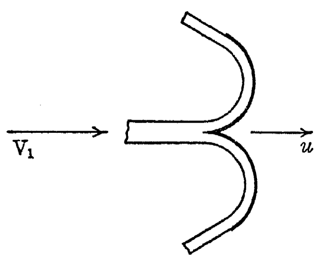


FIG. 160

$$F_x = \frac{W}{g}(V_1 - u)(\cos \theta - 1) \tag{176}$$

$$F_y = \frac{W}{g}(V_1 - u) \sin \theta \tag{177}$$

The force, F_y, is normal to the direction of motion of the vanes and it must be carried as a thrust on the bearings. It is often desirable to eliminate this thrust and it can be eliminated by the use of split vanes comparable to the one illustrated in Fig. 160. This type of vane is used on the

Pelton water wheel. A somewhat similar arrangement can be used in the impeller of a centrifugal pump. In the case of the pump, the liquid would probably enter axially from the ends of the impeller and pass radially out at the center. For split vanes, such as these, the normal force for one side of the vane would just balance that from the other half and the thrust on the bearings would be eliminated.

(*c*) *Effect of friction on the vane.* The force equations which have been developed in this article depend upon the assumption that the friction on the face of the vane is negligible. For cases in which friction were important, v_2 would be less than v_1, and the vector diagram for the conditions at the outlet would be used for determining the final velocity, V_2. The method of solution will be illustrated in the following example:

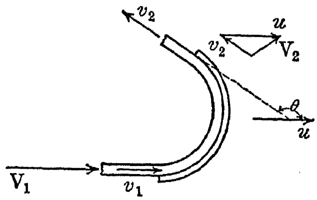


FIG. 161

Illustrative Problem: A jet of water having a cross-sectional area of 6 sq. in. and having a velocity of 160 ft. per sec. strikes the vane illustrated in Fig. 161. The vane moves in the direction of the jet at the rate of 70 ft. per sec. and has an angle $\theta = 120°$. The friction on the face of the vane is such that $v_2 = 0.8v_1$. Find the components of the force exerted by the vane on the jet.

$$W' = 62.4 \times \frac{6}{144} \times (160 - 70)$$

$$= 234 \text{ lb. per sec.}$$

$$v_1 = 160 - 70 = 90 \text{ ft. per sec.}$$

$$v_2 = 0.8 \times 90 = 72 \text{ ft. per sec.}$$

$$V_{2_x} = u + v_2 \cos \theta$$

$$= 70 - 72(0.5) = 34 \text{ ft. per sec.}$$

$$V_{2_y} = v_2 \sin \theta$$

$$= 72 \times 0.866 = 62.3 \text{ ft. per sec.}$$

$$\Delta V_x = V_{2_x} - V_1$$

$$= 34 - 160 = -126 \text{ ft. per sec.}$$

$$\Delta V_y = V_{2_y} = 62.3 \text{ ft. per sec.}$$

$$F_x = \frac{W'}{g} \Delta V_x$$

$$= \frac{234}{g} \times (-126) = -915 \text{ lb.} \qquad \textit{Ans.}$$

The minus sign indicates that the force acts opposite to the initial direction of the jet.

$$F_y = \frac{W'}{g}\Delta V_y$$

$$= \frac{234}{g} \times 62.3 = 453 \text{ lb.} \qquad \textit{Ans.}$$

Without friction, the value of the x-component of the force would have been

$$F_x = \frac{234}{g} \times (-135) = -980 \text{ lb.}$$

which shows the desirability for the surface of such vanes to be well polished in order that the friction might be low.

115. Power Developed on Moving Vanes. — The cases of the single vane and of the series of vanes will both be considered.

(*a*) *Power developed on a single vane with no friction.* Since power is the product of force and velocity, the power developed by the fluid on the vane is

$$\text{Power} = \frac{Wu}{gV_1}(V_1 - u)^2(1 - \cos\theta) \tag{178}$$

The sign has changed from that used in Eq. (174) since the work done by the water on the vane is now being considered. The initial kinetic energy in the jet in foot-pounds per second is

$$\text{K.E.} = \frac{W}{2g}V_1^2$$

$$\text{Efficiency} = \frac{2u(V_1 - u)^2(1 - \cos\theta)}{V_1^3} \tag{179}$$

In order to obtain the velocity for the maximum efficiency, the first derivative of Eq. (179) is set equal to zero.

$$-4u(V_1 - u) + 2(V_1 - u)^2 = 0$$

from which

$$u = V_1 \quad \text{or} \quad u = \frac{V_1}{3}$$

For the condition $u = V_1$, no force would be exerted by the jet on the vane and there would be a zero, or minimum, efficiency. If $u = V_1/3$, the

efficiency would be a maximum, the value of which can be found from Eq. (179).

$$\text{Max. Eff.} = \frac{8}{27}(1 - \cos\theta)$$

This has its maximum value when $\theta = 180°$, for which the efficiency is $\frac{16}{27}$ or 59 per cent.

(*b*) *Power developed on a series of vanes with no friction.*

$$\text{Power} = \frac{Wu}{g}(V_1 - u)(1 - \cos\theta) \tag{180}$$

$$\text{Initial K.E.} = \frac{W}{2g}V_1^2$$

$$\text{Efficiency} = \frac{2u}{V_1^2}(V_1 - u)(1 - \cos\theta) \tag{181}$$

The first derivative of Eq. (181) is now equated to zero

$$(V_1 - u) - u = 0$$

from which

$$u = \frac{V_1}{2}$$

Substituting this value of u in Eq. (181), it follows that the

$$\text{Max. Eff.} = 0.5(1 - \cos\theta) \tag{182}$$

The maximum value of the efficiency will occur when $\theta = 180°$, for which the efficiency would be unity. For this condition, $V_2 = 0$. Perfect efficiency is impossible for two reasons. No vane can be made frictionless, and some energy will be consumed in overcoming the friction. Secondly, a series of vanes cannot deflect the jet through an angle of 180° as the discharge from each vane interferes with the vane immediately following. An angle less than 180° must be used in order that the jet will clear the path of the next vane. For angles less than 180°, the absolute velocity of the jet is not zero.

The reader must not confuse the efficiencies given by equations (179) and (181) with the overall efficiencies of the machines concerned. The machine efficiency would also include the loss in the jet, windage loss, mechanical friction, loss due to possible leakage, etc. For a well designed impulse water wheel, overall efficiencies above 85 per cent have been attained.

PROBLEMS

250. A jet of water having an area of 3 sq. in. and a velocity of 130 ft. per sec. strikes a frictionless vane and is deflected through an angle of 150°. The vane is moving away from the jet with a velocity of 60 ft. per sec. Find the component of the force exerted by the jet on the vane in the direction of motion.

251. Suppose that the vane described in Prob. 250 is moving towards the jet with a velocity of 40 ft. per sec. Find the component of the force exerted by the jet on the vane in the direction of motion of the jet. *Ans.* $F = 2180$ lb.

252. A 2 in. jet of water is directed against a frictionless curved vane and is deflected through an angle of 175°. The vane moves in the direction of the jet with a velocity of 110 ft. per sec. The discharge is 6 c.f.s. Find the component of the force on the vane in the direction of motion. Find the horsepower developed and the water efficiency.

253. Let the conditions be the same as given in Prob. 252 except that the friction on the face of the vane reduces the relative velocity of the water with respect to the vane to 155 ft. per sec. Find the force exerted in the direction of motion, the horsepower developed and the water efficiency.

254. A paddle type of water wheel, 8 ft. in diameter, consists of flat plates set radially around the periphery of a wheel. The wheel is driven by a jet which issues from a rectangular orifice with a velocity of 24 ft. per sec. The jet is 24 in. wide and 3 in. high and strikes a wheel having a 42 in. radius. The velocity of the paddles is 10 ft. per sec. Find the speed in r.p.m. and the horsepower of the wheel.

Ans. $N = 27.2$ r.p.m., Power $= 5.91$ h.p.

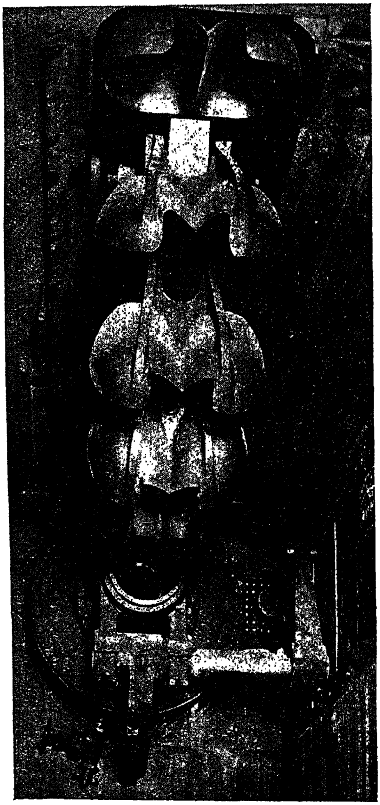


Fig. 162. View of impulse wheel runner from bottom showing nozzle outlet.

(*Courtesy Pelton Water Wheel Company.*)

This wheel is on a double over hung unit which develops 36,000 H.P. installed in the Tiger River Power House of the Pacific Gas and Electric Company. Maximum head 1221 ft., net operating speed 225 R.P.M. The wheel consists of 18-9½ in. buckets mounted on a 129½ in. pitch diameter.

116. The Impulse Wheel.—The impulse, or Pelton wheel utilizes the impulse of a jet of water in its operation. Such a wheel is shown in Fig. 162. The jet issues horizontally from a nozzle, the discharge from which is controlled by a needle valve. The jet strikes the ellipsoidal shaped buckets and is deflected

through an angle of about 165° to 170°, the exact angle varying somewhat with the design of the bucket and with the head on the nozzle. The jet is normally assumed to strike the buckets in a direction tangent to the circle whose radius extends to the centerline of the buckets and that the bucket is moving in the same direction as the jet. This condition is not satisfied as the bucket must cut into the path of the jet somewhat before reaching the lowest point in its travel.

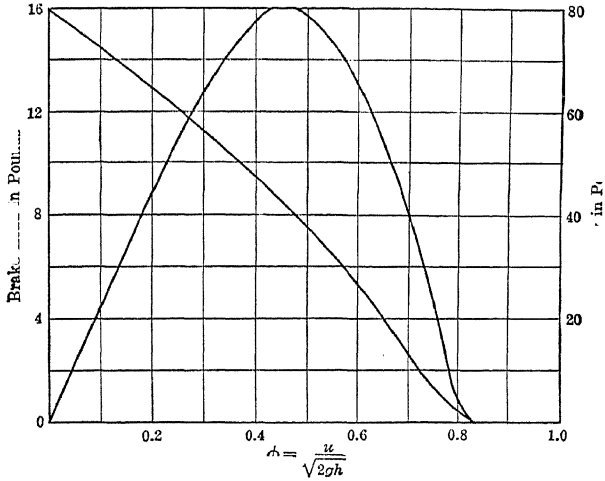


FIG. 163. Brake load efficiency curves for $10\frac{1}{2}$ in. impulse wheel.

For a given discharge and velocity of the jet, the force, or torque, exerted on the wheel depends upon the speed of the wheel. The maximum force is exerted at zero speed and then decreases as the speed increases. The runaway speed is attained when just enough force is developed to overcome the windage and frictional losses. The power output of the wheel is zero at both zero and runaway speeds and is a maximum for some intermediate speed. The power output, and efficiency, of the wheel reaches a maximum when the ratio of the peripheral velocity of the wheel at the centerline of the buckets is about 0.43 to 0.48 of the theoretical spouting velocity of the water in the jet with losses neglected. This condition is expressed by the equation

$$u_1 = \phi\sqrt{2gh} \tag{183}$$

where h is the total head at the base of the nozzle and ϕ is the *speed ratio* having the value given above for the most efficient operating conditions. The maximum value of ϕ for runaway speed is about 0.8. Typical operating curves for the impulse wheel are given in Fig. 163.

PROBLEMS

255. A nozzle for which $C_v = 0.98$ discharges a jet 4 in. in diameter under a head of 1250 ft. The jet acts upon a wheel which is 5 ft. in diameter and the jet is deflected 165°. Find the force exerted upon the buckets when $\phi = 0.44$. Neglect friction on the buckets.

256. A 3 in. jet having a velocity of 240 ft. per sec. acts on an impulse wheel whose buckets are curved 165°. If 5 per cent of the relative velocity is lost as the jet passes over the buckets, find the power when $\phi = 0.40$. *Ans.* 1097 h.p.

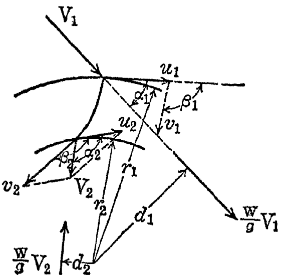


FIG. 164

117. Dynamic Action in Rotating Channels. — When flow occurs in a rotating channel, this channel would probably be a passage in a turbine runner or in a pump impeller. For the turbine, work would be furnished by the fluid as it passed through the runner. In the pump, work would be done on the fluid by the runner and it would pass out of the impeller possessing more energy than when it entered.

The condition of inward flow through a turbine runner is shown in Fig. 164. This would be a pressureless turbine, such as the Girard turbine which was used in this country some years ago and which may still be found in use in Europe. The force exerted on the fluid by the vanes would be

$$F = \frac{W}{g}(\Delta V) = \frac{W}{g}(V_2 \rightarrow V_1)$$

and the force exerted by the fluid on the vanes would be equal to this but would act in the opposite direction

$$F_w = \frac{W}{g}(V_1 \rightarrow V_2) \tag{184}$$

Equation (184) can be written

$$F_w = \frac{W}{g}V_1 \rightarrow \frac{W}{g}V_2$$

which is equivalent to a force at the inlet to the runner and another at the outlet. The torque exerted is

$$T = \frac{W}{g}V_1 d_1 - \frac{W}{g}V_2 d_2$$

$$= \frac{W}{g}(V_1 r_1 \cos\alpha_1 - V_2 r_2 \cos\alpha_2) \tag{185}$$

The horsepower of the water would be

$$\text{H.P.} = \frac{T\omega}{550}$$

where

$$\omega = \frac{u_1}{r_1}$$

$$\text{H.P.} = \frac{Wu_1}{550\, gr_1}(V_1r_1 \cos \alpha_1 - V_2r_2 \cos \alpha_2) \tag{186}$$

Since H.P. $= WH/550$, and remembering that $\omega = u_1/r_1 = u_2/r_2$, the head utilized by the turbine would be

$$H = \frac{V_1u_1 \cos \alpha_1 - V_2u_2 \cos \alpha_2}{g} \tag{187}$$

In the turbine, β_1 should have such a value that the relative velocity of the water with respect to the vane, v_1, would be tangent to the direction of the tip of the vane. For this condition, there would be no shock at entrance. The value of α_2 should preferably be 90°. Much better operating conditions exist when the water leaves the turbine with no circumferential component. This component produces undesirable vortex action in the *draft tube*, which is the tube conducting the water away from the turbine.

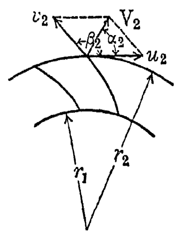


FIG. 165

The pump is illustrated in Fig. 165. The vector diagram of the velocities has been drawn only for the outlet conditions since any momentum possessed by the liquid at the suction side of the pump was produced by the pump impeller. It is equivalent to a condition of no initial momentum. In this case, the torque would be

$$T = \frac{W}{g} V_2r_2 \cos \alpha_2 \tag{188}$$

and the power imparted to the fluid would be

$$\text{H.P.} = \frac{Wu_2}{550g} V_2 \cos \alpha_2 \tag{189}$$

PROBLEMS

257. The absolute velocity of water entering a turbine is 75 ft. per sec., and that leaving is 15 ft. per sec. $\alpha_1 = 30°$, $\alpha_2 = 85°$, $r_1 = 2.5$ ft., $r_2 = 2.0$ ft. The discharge is 20 c.f.s. (*a*) Find the torque exerted on the wheel. (*b*) If the speed of the wheel is 240 r.p.m., find the power delivered to the wheel.

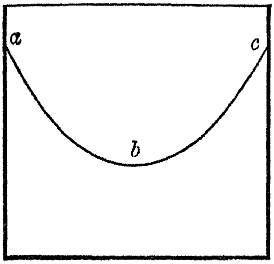
258. Find the head utilized by the wheel in Prob. 257.

259. A propeller type fan which is 30 in. in diameter delivers 7500 cu. ft. of air per min. The temperature is 85° F. and the barometer reads 29 in. of Hg. $R = 53.34$. (*a*) Find the thrust on the fan. (*b*) Find the power required to drive the fan if it has an efficiency of 60 per cent. *Ans.* (*a*) $F = 6.95$ lb.; (*b*) 0.54 h.p.

CHAPTER XI

CENTRIFUGAL PUMPS

118. Description and Classification. — It was demonstrated in Art. 44 and again in Art. 117 that the potential energy which a liquid possessed could be changed by forcing the liquid to move in a curved path. Imagine a circular container which is partially filled with water. Let the water be rotated in the container by means of a paddle wheel until the particles of water are moving with a constant angular velocity. The particles near the circumference of the container will be moving with the highest velocity and the free surface will be at a higher elevation near the rim than at the



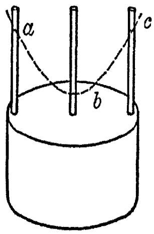


FIG. 166

center. This condition is illustrated in Fig. 166*a*. Now, if a lid were placed on the container and it was filled with liquid which was forced to rotate as in the previous example, the pressure would be greatest near the circumference. Should a series of piezometer tubes be placed along a diameter, the liquid levels in these tubes would define a curve similar to *abc* in Fig. 166*b*. Openings could be made in the periphery of the container and the liquid discharged would possess more energy than that possessed by an equal volume of liquid at the center of the container.

Should provision be provided for admitting liquid at the center and for discharging it at the periphery, the enclosed container with the paddle wheel for producing the forced rotation would constitute a primitive type of centrifugal pump. The efficiency would be very low since there would be excessive shock losses and there would be no provision for converting the kinetic energy of the discharging liquid into pressure energy. In order to minimize shock losses, curved vanes would be used in the impeller and a

case, having a receiving chamber of gradually increasing size, would be used to convert a portion of the kinetic energy into pressure energy.

The modern centrifugal pump contains an impeller which is mounted on a shaft. The impeller rotates within the casing and takes liquid, having a low pressure, in through the *eye* of the impeller and discharges it out into the case with a considerably higher head. From the case, the liquid flows out through the discharge nozzle, which is the name of the discharge open-

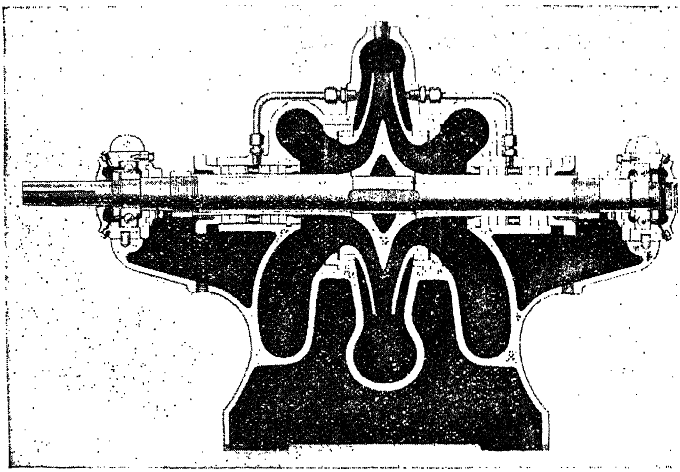


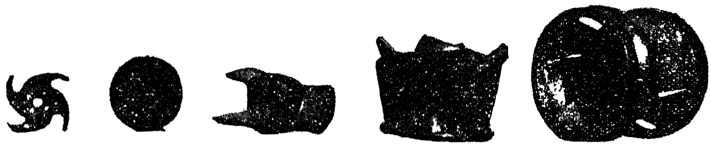
FIG. 167. Sectional view of double suction centrifugal pump. (Courtesy Fairbanks-Morse Co.)

ing of the pump. A sectional view of a medium head, rather high capacity centrifugal pump is shown in Fig. 167. The essential parts of the pump can be clearly seen. Note the smooth curves along which the liquid flows in passing through the impeller.

Pumps having rotating impellers are normally called centrifugal pumps. These include the pumps which force the liquid outward, thus causing an increase in the velocity head of the liquid due to the higher peripheral velocity, and also those having a propeller type of impeller which forces the liquid in an axial direction. The axial flow pump is not a true centrifugal pump.

The most important class of centrifugal pumps is the volute type similar to that illustrated in Fig. 167. This pump has a casing, made in the form of a spiral or volute, which contains either the open or closed impeller such as illustrated in Figs. 168*a* and 168*b*. The open impeller is somewhat less costly to manufacture and is used in the lower priced pumps. The leakage

past the sides of the vanes from the discharge back into the suction is rather large resulting in a low efficiency, but there is little tendency for the impeller to clog with foreign matter, and its life is longer than that of the closed impeller. The closed impeller, which has the shrouds on the sides of the vanes, Figs. 167 and 168*b*, presents a smooth, well defined path for the liquid and permits less leakage from the discharge of the impeller back past it into the suction. These improvements result in an improved efficiency. The impeller may be either single or double suction, depending upon whether the suction is from one, or both sides of the impeller. The



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 168. Types of centrifugal pump impellers.

impellers shown in Figs. 168*a* and 168*b* are single suction, while Figs. 167 and 168*e* are double suction. The capacity for a given diameter and rotative speed is larger for the double suction impeller than it is for the single. One double suction impeller is roughly equivalent to two single suction ones placed back to back. There is a sizable thrust present in the single suction impeller which is eliminated by the balancing which is provided in the double suction type.

When the head which the pump must furnish exceeds about 300 ft., a pump having more than one impeller on the same shaft is commonly used with the impellers acting in series. A two-stage pump having the impellers placed back to back is shown in Fig. 169. This arrangement eliminates the hydraulic thrust and makes for very desirable operating conditions.

The trash pump, which is illustrated in Fig. 170, is also of the volute type. In order to insure against clogging of the impeller, the number of vanes has been reduced and the distance between the shrouds increased. In fact, the pump illustrated in Fig. 170 has been so designed that it will pass any spherical body which would go through a pipe one pipe-size smaller than that of the intake pipe.

A centrifugal pump having diffusion vanes in the case between the outlet of the impeller and the volute is known as a turbine pump. It is the purpose of these vanes to more efficiently convert the velocity head of the liquid into pressure head, but their presence tends to require a pump having a case of large diameter and one of higher initial cost. The present designers are capable of producing a volute type pump of comparable efficiency to that of the horizontal type of turbine pump. In fact, the

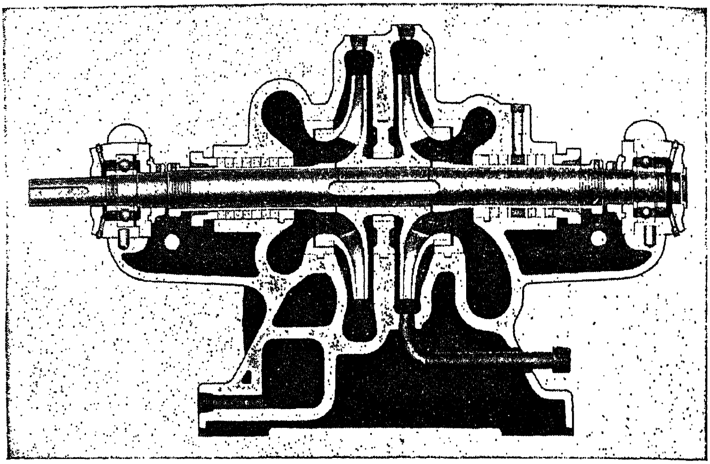


FIG. 169. Sectional view of two-stage centrifugal pump. (Courtesy Fairbanks-Morse Co.)

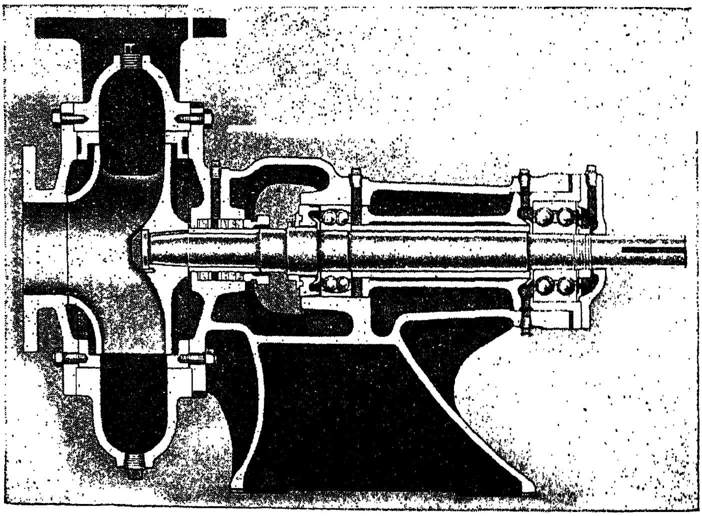


FIG. 170. Sectional view of trash pump. (Courtesy Fairbanks-Morse Co.)

efficiencies of some volute pumps exceed those of turbine pumps designed for similar operating conditions. It seems that there is little reason for choosing a turbine pump in preference to a well designed volute pump for the horizontal setting and few are being built at the present time. The turbine type is much used for deep well purposes because a good efficiency cannot be obtained from the volute type which has been designed to meet the space limitations.

The impeller of the axial flow pump, see Fig. 168*c*, is similar to a propeller. It changes the head by producing an axial thrust. There is no radial action in the impeller and no centrifugal head is transmitted to the liquid. This pump is used on low lifts where large volumes are needed.

An impeller possessing the combined traits of the volute and axial flow types is known as the mixed flow impeller. Such an impeller appears as Fig. 168*d*. The volume which it is capable of discharging is between that of the volute and axial flow types. The efficiencies of the mixed flow and axial flow pumps are somewhat less than those of the volute types for a given discharge capacity.

Centrifugal pumps are used for all kinds of liquids and mixtures, such as oils, wood pulp, water, sand-water mixtures, sewage, sludge, and dredging materials with the solid particles ranging up to the size of large gravel. Certain problems encountered in handling this wide range of materials will be considered in later articles.

119. Centrifugal Action and Losses in Pumps. — In Art. 117, the torque needed to force a liquid through the rotating passages of a pump was found to be

$$T = \frac{W}{g} V_2 r_2 \cos \alpha_2 \tag{188}$$

and the power required to produce this torque was given by the equation

$$\text{H.P.} = \frac{W u_2}{550 g} V_2 \cos \alpha_2 \tag{189}$$

Equation (189) is the water horsepower needed to force the liquid through the vanes of the pump. Since H.P. $= WH/550$, the head produced in the impeller would be

$$H = \frac{V_2 u_2}{g} \cos \alpha_2 \tag{190}$$

Now the head actually developed by the pump is less than that developed by the impeller due to various losses. These losses exist in the form of

(1) Friction and eddies within the impeller.

(2) Shock losses due to the fact that the liquid has a high velocity upon leaving the passages in the impeller and this cannot be reduced to the

lower velocity of the liquid in the case without an accompanying loss. Shock losses may also occur at the eye of the impeller.

(3) Frictional loss after the liquid leaves the impeller until it reaches the discharge nozzle of the pump.

The discharge of the pump is less than the discharge of the impeller due to leakage taking place from the discharge side of the impeller along the side of the impeller back into the suction. This flow is much greater for the open impeller than for the closed and is greater after wear takes place. For low discharges, there may also be a circulation of liquid within the passages of the impeller itself.

The mechanical friction consists mainly of disc friction due to the drag on the liquid in the clearance spaces. In addition to this, there is the friction occasioned by the packing in the stuffing boxes, and the bearing friction.

For a constant speed, the mechanical friction is fundamentally constant. The leakage loss is proportional to the head developed by the impeller. Referring to Eq. (190), it can be seen that the head is proportional to V_2, or to the discharge itself. The fluid friction and eddy losses in any system for turbulent flow vary approximately as the square of the velocity, or the square of the discharge. It can then be said that, for a constant speed, the developed head can be approximately expressed as

$$H = A - BQ - CQ^2 \tag{191}$$

The constant, B, may be either positive or negative depending upon the design of the pump. It can then be seen that the overall head of the pump may be a maximum for zero discharge and then drop as the flow increases, or it may rise for a time and then decrease for the larger discharges as the influence of the last term becomes more effective. Thus, we have pumps with falling or rising characteristics, the discussion of which appears in the following article.

120. Pump Characteristics. — The head produced by a centrifugal pump operating at a certain speed is a function of the discharge. The curve showing the relationship between the head developed by the pump and the discharge is known as the pump characteristic. On this same sheet, curves are generally added to show the brake-horsepower-discharge, and the efficiency-discharge relationships. Such curves appear in Fig. 171. The curves in this figure are for a pump having a falling characteristic. The maximum capacity of this pump for a speed of 1750 r.p.m. is 2340 g.p.m., but the pump would not be rated at this capacity. Referring to the efficiency curve, it is seen to have a maximum value of 86.2 per cent for a discharge of 1930 g.p.m., but that it drops little for some increase in discharge. This pump is rated as a 2000 g.p.m. pump. At this discharge,

the developed head is slightly in excess of 51 ft. This same pump could be used as a 1500 g.p.m. pump for a head of 67 ft. It is interesting to note how nearly constant the brake-horsepower remains for this pump. For

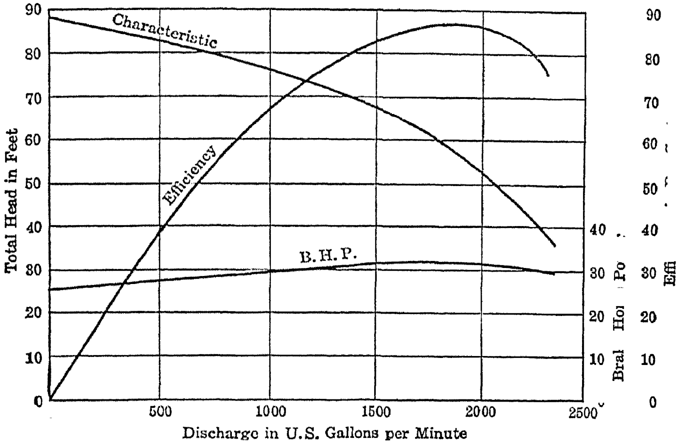


FIG. 171. Characteristic curves for centrifugal pump having $9\frac{1}{4}$ in. impeller and a speed of 1750 r.p.m.

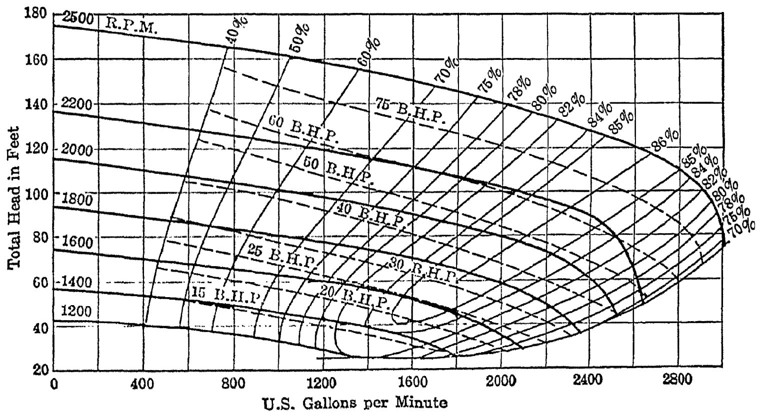


FIG. 172. General characteristic curves for centrifugal pump having $9\frac{1}{4}$ in. impeller. (Courtesy Fairbanks-Morse Co.)

pumps having the falling characteristics, there is little danger of overloading the motor. A 30 h.p. motor would be used.

A better picture concerning the operation of any pump is afforded by the general characteristic curves such as appear in Fig. 172. These curves are for the same pump considered in Fig. 171. A series of iso-efficiency

curves are now drawn and the efficient operating range of the pump can easily be seen. The efficient operating range limits the choice of the pump to requirements which it can effectively meet.

121. Variations in the Speed and the Diameter of the Impeller. — The Homologous Series. — For a given pump, or series of pumps, there are three changes which can be made. Different characteristics can be obtained by operating the same pump at different speeds. The effect of this was shown in Fig. 172. It might be desired to pump a smaller quantity than the normal rated output of the pump, but more than would be possible if the pump were driven by a motor having a slower speed. This can be accomplished by turning the impeller down in a lathe and thus obtaining a smaller peripheral velocity for a given speed of rotation. This would result in a smaller discharge at a lower head and there would be little loss in efficiency. If a major change in capacity were called for, it could be obtained by the manufacture of a pump of different size. Should all of the dimensions of the new pump be changed until they are proportional to those of the original pump, and should a range of such pumps be manufactured, they would constitute a homologous series. The effect of these various changes will now be considered.

(*a*) *Effect of change in speed.* In order for the efficiency to remain high, the direction of the various vectors illustrated in Fig. 165 should remain unaltered. Since u_2 is proportional to the rotative speed, N, v_2 would also be proportional to N. Since no change has been made in the area of the water passages, the discharge, Q, would be proportional to v_2, or N. As shown in Eq. (190), the head is proportional to the product of V_2 and u_2. Since these are both proportional to N, the head is proportional to N^2. The power is proportional to QH, and would be proportional to N^3. These relations can be expressed as

$$Q = k_1 N \tag{192}$$

$$H = k_2 N^2 \tag{193}$$

$$\text{H.P.} = k_3 N^3 \tag{194}$$

(*b*) *Effect of change in diameter of impeller for same casing.* The vector relationship must remain the same, as in the preceding case. Since u_2 is proportional to the diameter, d, v_2 is proportional to d. The width of the water passage will be assumed not to have changed, but the circumferential dimension is proportional to d. The discharge, Q, is then proportional to d^2. Since V_2 and u_2 are both proportional to d, the head is proportional to d^2. The power would be proportional to $d.^4$ These are expressed as

$$Q = k_4 d^2 \tag{195}$$

$$H = k_5 d^2 \tag{196}$$

$$\text{H.P.} = k_6 d^4 \tag{197}$$

(*c*) *The homologous series.* In this series, the dimensions of the corresponding parts are all proportional to the diameter, d. For constant speed conditions, the various velocities are proportional to the diameter, but the area of the passages is now changed in both directions and is proportional to d^2. The discharge, Q, is therefore proportional to d^3. The head is proportional to d^2 and the horsepower is proportional to d^5.

The above relationships can be combined with those expressed in equations (192), (193), and (194) to produce the complete expressions.

$$Q = k_7 d^3 N \tag{198}$$

$$H = k_8 d^2 N^2 \tag{199}$$

$$\text{H.P.} = k_9 d^5 N^3 \tag{200}$$

The equations which have been developed in this article apply only approximately due to the fact that the characteristics of the pump are not functions of the size, speed, and angles of the impeller alone. Such factors as leakage, disc friction, bearing friction, etc., have not been considered. In general, the efficiency of a pump increases somewhat with an increase in speed. The efficiency of the larger pumps in a homologous series is also greater than that for the smaller sizes. Based on experience, the manufacturers know approximately the discrepancy that can be expected and can closely predict the performance of a second pump if that of a similar one is known for another condition. The characteristic curve for a speed of 1150 r.p.m. has been computed for the pump in Fig. 171, and the actual characteristic has been added for comparative purposes. This comparison appears as Fig. 173.

PROBLEMS

260. A given pump delivers 2200 g.p.m. against a head of 60 ft. when operating at 1750 r.p.m. and requires 40 h.p. to drive it. Find the discharge, head and power for this pump when run at 1150 r.p.m.

261. A 6 in. pump running at 1800 r.p.m. delivers 1000 g.p.m. against an 80 ft. head and requires 29 h.p. to drive it. Determine the discharge, head and power for this pump when run at 2000 r.p.m. for homologous operation.

262. Determine the discharge, head and power of an 8 in. pump, similar to the above 6 in. pump, run at 1500 r.p.m. for homologous operation.

Ans. $Q = 1975$ g.p.m.; $H = 98.7$ ft.; Power = 70.7 h.p.

263. Determine the most efficient discharge, head and power of a pump having a 12 in. diameter impeller, similar to that whose characteristics are shown in Fig. 172, run at 1650 r.p.m. for homologous operation.

122. Conditions in Service. — The centrifugal pump operates efficiently only when rotating with a high peripheral speed. This high speed is easily attained, even with small diameters, by direct connection to an electric motor or steam turbine. For a given output, the high speed motor is smaller in size and less expensive than the slower speed one. This tends

towards a low first cost. The motor is quite flexible in that it can be purchased to run at one of a number of different standard speeds and a pump can be chosen to meet these speed conditions.

The steam turbine is also an efficient prime mover when operating at high speeds. It is considerably more flexible in that the speed of the same machine can be made to vary by throttling, changing the governor setting

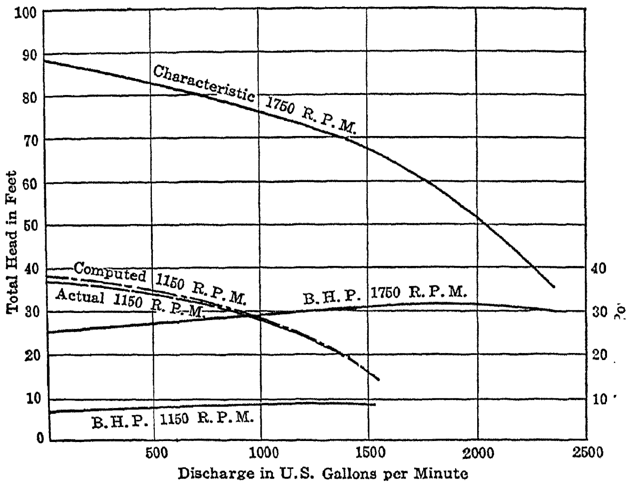


FIG. 173. Effect of change of speed on characteristic curve of centrifugal pump having $9\frac{1}{4}$ in. impeller.

or by changing the steam pressure. The steam turbine is an especially good prime mover should it be desired to maintain a constant pressure in the line for widely fluctuating discharges. By having the governor actuated by the pressure in the discharge line, the speed of the turbine would increase with a tendency for a drop in pressure and the larger discharge would be furnished without a serious drop in pressure. In this way, the pump could be made to operate at efficient speeds even for widely varying load conditions.

According to Standards of Hydraulic Institute, membership in which is maintained by the various pump manufacturers, a pump is tested with a 15 ft. suction head while pumping water. This does not mean that higher suction lifts cannot be maintained, but it is advisable that the suction lift should never exceed 22 ft. when cold water is being pumped. Warm water could never be pumped successfully with this great suction lift. For high suction lifts, there is a tendency for the liquid to vaporize and for cavitation to occur at the eye of the impeller. The centrifugal pump will not handle

gases and vapors, and the presence of a small amount of either seriously reduces the capacity of the pump if it does not cause it to lose its prime completely. High suctions may also cause serious vibrations.

When the centrifugal pump is used for pumping volatile liquids, or hot water, very low suction lifts must be used. It may even be necessary for a positive pressure to be used on the suction side to force the liquid into the pump. This may be accomplished by placing the pump in a pit below the level of the liquid in the sump.

When a centrifugal pump fails to operate in a satisfactory manner, the trouble is generally on the suction side. In many cases, air is leaking into the suction line. This might be due to air escaping from an air pocket in the suction line, to the use of bell and spigot joints on the suction line, which should never be used, to a suction lift which is too great, or to leaky packing about the shaft of the pump. In some cases, the liquid carries dissolved gases which are released under the low pressure conditions. When a centrifugal stops pumping, it should be promptly stopped, reprimed, and then started again. The quantity of gas removed may be sufficient to permit it to operate in a satisfactory manner. No centrifugal pump will operate without having first been primed. There are a number of methods used in priming. Probably the most common is either by use of a check valve, known as a foot valve, on the end of the suction pipe which permits the pump to be completely filled from the discharge tank, or by use of an ejector which would be attached to the top of the case. The ejector may operate from either a high pressure water, air, or steam line. In the case of large size pumps, a small auxiliary pump is often used for priming purposes. Regardless of the procedure, it is essential for the suction line and pump case to be completely freed of air.

Where the pump is to be used for pumping corrosive liquids, special linings are used in the pump and an impeller, made of a corrosion resisting material, is installed. When liquids containing much abrasive material are to be pumped, the case and impeller are made of alloys which can better resist the abrasion. At the present time, pumps for handling very abrasive materials, such as would be encountered in mines or gravel pits, are sometimes lined with rubber. The entire case and impeller are lined, and the life of the pump has been increased several times.

In a pump installation, the required discharge normally varies considerably. Let it be assumed that the pressure at the discharge end of the piping system is to remain constant. The head needed at the pump can then be expressed by the equation

$$H = A + BQ^2 \tag{201}$$

where A is the sum of the static lift and the required delivery head and

BQ^2 represents the frictional losses in the piping system. Since the head delivered by a pump normally increases as the discharge decreases, the conditions are conflicting and the constant delivery pressure can only be maintained by the use of a throttling valve located somewhere in the discharge line. Such a condition would be costly in power consumed and could better be accomplished by means of a variable speed drive. For the direct connected electric motor drive, this can only be accomplished by means of a multi-speed motor. For a steam turbine drive, some variation in speed can be obtained by means of throttling in the steam line. An example will now be solved to determine the advantage of a two-speed installation.

Illustrative Problem: The daily load requirements for the pump, whose characteristics are given in Fig. 173, require that it furnish 2000 g.p.m. for 4 hrs., 1200 g.p.m. for 6 hrs. and 800 g.p.m. for 4 hrs. Find the saving per year if a two-speed (1750 r.p.m. and 1150 r.p.m.) motor is used instead of a 1750 r.p.m. single speed motor. Cost of electricity $1\frac{1}{2}$ cents per K.W.H. Assume a motor efficiency of 92 per cent for the full load and 85 per cent for the part load condition.

Constant speed condition.

B.H.P. for 2000 g.p.m. = 31

$$\text{K.W.H.} = \frac{31 \times 0.746 \times 4}{0.92} = 100.6$$

B.H.P. for 1200 g.p.m. = 29.7

$$\text{K.W.H.} = \frac{29.7 \times 0.746 \times 6}{0.92} = 144.5$$

B.H.P. for 800 g.p.m. = 28.4

$$\text{K.W.H.} = \frac{28.4 \times 0.746 \times 4}{0.92} = 92.1$$

K.W.H. per day = 337.2

K.W.H. per year = 123,100

Cost of power per year = \$1846.50

Two-speed operation.

B.H.P. for 2000 g.p.m. = 31

$$\text{K.W.H.} = \frac{31 \times 0.746 \times 4}{0.92} = 100.6$$

B.H.P. for 1200 g.p.m. = 8.8

$$\text{K.W.H.} = \frac{8.8 \times 0.746 \times 6}{0.85} = 46.3$$

B.H.P. for 800 g.p.m. = 8.3

$$\text{K.W.H.} = \frac{8.3 \times 0.746 \times 4}{0.85} = 29.2$$

K.W.H. per day = 176.1

K.W.H. per year = 64,300

Cost of power per year = \$964.50

The savings per year made possible by the two-speed installation amount to \$882.00. If depreciation and interest charges amount to 15 per cent annually, the value of these savings would be \$5880. It would be advisable to make the two-speed installation if the additional cost were less than \$5880.

(*Note:* The actual increased cost for the two-speed installation would be about \$300.00 for these capacities.)

PROBLEMS

264. A pump whose maximum capacity is 1100 g.p.m. has a suction pipe consisting of 50 ft. of 6 in. clean steel pipe and one medium radius ell. The end of the suction pipe projects into the reservoir. Find the elevation of the pump above the water surface in the sump if the maximum vacuum is 16 ft. of water. $T = 70°$ F.

265. The pump, whose general characteristics curve are shown in Fig. 172, is driven by a steam turbine at a speed of 1750 r.p.m. for a discharge of 2000 g.p.m. The governor varies the speed in order to maintain a constant pump head as the discharge changes from 1000 g.p.m. to 2800 g.p.m. Find the speed, efficiency and power for the three discharges.

266. The pump whose characteristic curves are shown in Fig. 173 is required to deliver 2100 g.p.m. for 5 hrs., 1500 g.p.m. for 2 hrs., 1100 g.p.m. for 2 hrs. and 700 g.p.m. for 1 hr. each day. The cost of electric power is 1.7 cents per K.W.H. and the motor efficiency for this range is given by the equation $e = 94 - 15 \left(\frac{40 - \text{B.H.P.}}{40}\right)$ per cent. (*a*) Find the annual cost for single speed operation at 1750 r.p.m. (*b*) Find the annual cost for two-speed operation at 1750 r.p.m. and 1150 r.p.m. (*c*) Find the additional investment that would be justified for the two-speed system based on a 20 per cent interest and depreciation charge.

123. Pumping Liquids with Specific Gravities Differing from Unity. —

When selecting a pump for liquids either lighter or heavier than water, care should be exercised since both the head and power requirements are affected. This can best be illustrated by examples.

Illustrative Problem 1: Liquid heavier than water.

A pump is required to circulate a brine having a S.G. = 1.2 against a net pressure of 40 lb. per sq. in. at the rate of 300 g.p.m. Find the pumping head and required horsepower with a 60 per cent overall efficiency.

$$H = \frac{40 \times 144}{62.4 \times 1.2} = 77 \text{ ft.} \qquad \textit{Ans.}$$

$$\text{H.P.} = \frac{WH}{550 \times \text{eff.}} = \frac{8.33 \times 1.2 \times 300 \times 77}{550 \times 60 \times 0.60}$$

$$= 11.7 \qquad \textit{Ans.}$$

Illustrative Problem 2: Liquid lighter than water.

A pump is required to pump gasoline having a S.G. = 0.70 against a net pressure of 40 lb. per sq. in. at the rate of 300 g.p.m. Find the pumping head and

required horsepower with a 60 per cent overall efficiency.

$$H = \frac{40 \times 144}{62.4 \times 0.70} = 132 \text{ ft.} \qquad \textit{Ans.}$$

$$\text{H.P.} = \frac{WH}{550 \times \text{eff.}} \qquad \frac{8.33 \times 0.70 \times 300 \times 132}{550 \times 60 \times 0.60}$$

$$= 11.7 \qquad \textit{Ans.}$$

The motor requirements are the same in the two cases but the difference in the heads would require the use of different pumps.

PROBLEMS

267. Find the head and power required to pump kerosene, S.G. = 0.80, against a net pressure head of 50 lb. per sq. in. at the rate of 500 g.p.m. with an efficiency of 70 per cent.

268. A mixture of sand and water, having a specific gravity of 1.08, is pumped against a net head of 35 ft. at the rate of 3600 g.p.m. The efficiency is 78 per cent. Find the net pump pressure and the required power.

Ans. p = 16.4 lb. per sq. in., Power = 44.1 h.p.

124. Pumping of High Viscosity Liquids. — When a centrifugal pump is used for pumping liquids having high viscosities,[1] the capacity is decreased and the required power is increased. The decrease in capacity is less for a pump having wide passages than it is for a pump having narrow ones since the flow can take place more freely through the wide passages. The losses are relatively less in the low speed pumps than in the high speed ones. The power requirements increase mainly because of an enormous increase in the disc friction. Since the area subjected to disc friction is proportional to the square of the diameter of the impeller, only pumps having impellers with small diameters should be used. Pumps having impellers of small diameter automatically have the larger distances between the shrouds which is also advantageous from the standpoint of a small decrease in the capacity.

It is very difficult to define the viscosity of the liquid which is being pumped. The liquid will have one viscosity in the suction line. When it is in the pump, that portion of the liquid which is flowing through the passages of the impeller will have another viscosity, while that which is in the clearance spaces will have a higher temperature and a correspondingly lowered viscosity. Now if the pump is a multi-stage one, the problem is still further complicated because the liquid enters the later stages at a higher temperature and a lowered viscosity. This condition cannot increase the capacity of the later stages of the pump, but it does decrease

[1] M. D. Aisenstein, "Characteristics of Centrifugal Oil Pump," *T.A.S.C.E.*, V. 94, p. 425, 1930.

the power requirements of these stages in comparison to the requirements of the first stage.

Typical characteristic curves are shown in Fig. 174 for a centrifugal pump handling liquids having a wide range of viscosity. In this figure, the viscosities are given in Saybolt Seconds Universal and these can be

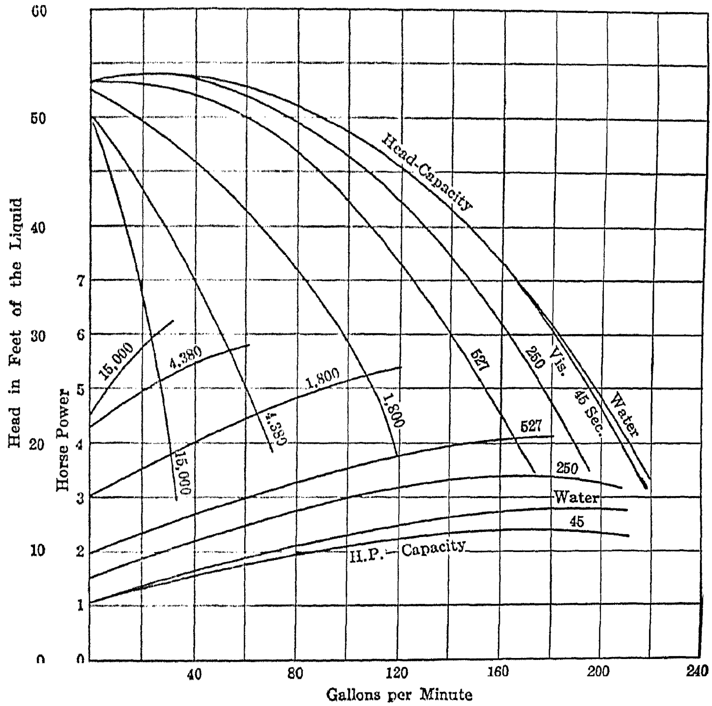


Fig. 174. Effect of viscosity on the head-capacity and horsepower-capacity of a Goulds single stage centrifugal pump operating at 1450 r.p.m. with oils of different viscosities. (Courtesy of Goulds Pumps, Inc.)

converted to kinematic viscosities if the reader so desires. The reader should note that the power requirements show a pronounced increase for the higher viscosities. The capacity of the pump does not decrease seriously so long as the viscosity does not exceed 250 S.s.u. Great care should be exercised in the choice of a pump for the highly viscous liquids since the capacity is much less than the water capacity while the power requirement is much greater than for water.

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